Testing graphs for colorability properties*

Eldar Fischer †

Abstract

Let P be a property of graphs. An ϵ-test for P is a randomized algorithm which, given the ability to make queries whether a desired pair of vertices of an input graph G with n vertices are adjacent or not, distinguishes, with high probability, between the case of G satisfying P and the case that it has to be modified by adding and removing more than $\epsilon\binom{n}{2}$ edges to make it satisfy P. The property P is called testable, if for every ϵ there exists an ϵ-test for P whose total number of queries is independent of the size of the input graph. Goldreich, Goldwasser and Ron [12] showed that certain graph properties, like k-colorability, admit an ϵ-test. In [2] a first step towards a logical characterization of the testable graph properties was made by proving that all first order properties of type "$\exists\forall$" are testable while there exist first order graph properties of type "$\forall\exists$" which are not testable. For proving the positive part, it was shown that all properties describable by a very general type of coloring problem are testable.

While this result is tight from the standpoint of first order expressions, further steps towards the characterization of the testable graph properties can be taken by considering the coloring problem instead. It is proven here that other classes of graph properties, describable by various generalizations of the coloring notion used in [2], are testable, showing that this approach can broaden the understanding of the nature of the testable graph properties. The proof combines some generalizations of the methods used in [2] with additional methods.

Keywords: property testing, graph coloring, regularity lemma.

1 Introduction

In all places where it is not stated otherwise, all graphs considered are finite, undirected, and have neither loops nor parallel edges. We also assume that all such graphs (as well as the other

*An extended abstract of this paper has appeared in the proceedings of the 12th ACM-SIAM SODA (2001), 873–882.

†Faculty of Computer Science, Technion – Israel Institute of Technology, Haifa, Israel. Email address: eldar@cs.technion.ac.il

combinatorial structures discussed in the following) have a labeled set of vertices. In what follows, we use the notation of [5] except where stated otherwise.

Let P be a property of graphs. A graph G with n vertices is called *ϵ-far from satisfying P* if no graph $\tilde{G}$ with the same vertex set, which differs from G in no more than $\epsilon\binom{n}{2}$ places (i.e. can be constructed from G by adding and removing no more than $\epsilon\binom{n}{2}$ edges), satisfies P. An *ϵ-test* for P is a randomized algorithm which, given the quantity n and the ability to make queries whether a desired pair of vertices of an input graph G with n vertices are adjacent or not, distinguishes with probability at least $\frac{2}{3}$ between the case of G satisfying P and the case of G being ϵ-far from satisfying P. The property P is called *testable*, if for every fixed $\epsilon > 0$ there exists an ϵ-test for P whose total number of queries is bounded by a function of ϵ (and is independent of the size of the input graph).

As to be expected, all properties discussed here are invariant with regard to graph isomorphisms. Since we restrict ourselves to tests with a constant number of queries, by [13] this allows us to assume without loss of generality that a given ϵ-test actually chooses a uniformly random constant size subset of the vertices of the input graph, queries about all vertex pairs in it, and gives the output based on the (queried upon) subgraph induced by the chosen vertex set.

The general notion of property testing was first formulated by Rubinfeld and Sudan [19], who were motivated mainly by its connection to the study of program checking. The study of this notion for combinatorial objects, and mainly for labeled graphs, was introduced by Goldreich, Goldwasser and Ron [12], who showed that several interesting graph properties, like k-colorability, are testable. The fact that k-colorability is testable was, in fact, already proven implicitly in [17] (see also [1]), using the Regularity Lemma of Szemerédi [20], but in the context of property testing it was first studied in [12]. Since then the subject of property testing in graphs and other combinatorial objects has rapidly progressed; see for example the surveys [8] and [18]. In [2] a first step in the direction of a logical characterization of the testable graph properties was taken, by proving the following.

Theorem 1.1 ([2]) *All first order properties of type "$\exists\forall$" are testable. On the other hand, there exists a first order property of type "$\forall\exists$" which is not testable.*

The definition of first order properties is explained in detail in [2]. Briefly, these are properties that can be formulated by first order expressions about graphs, that is, expressions that contain quantifiers (over vertices), Boolean connectives, equality and adjacency. The topic of logical characterizations of testable properties has been investigated also for other combinatorial objects, e.g. in [3], [16] and [10] for properties of Boolean strings and in [9] for properties of matrices.

In order to prove the positive part of Theorem 1.1 the following notion of (vertex) colorability was defined. Note that this is a generalization of the previously investigated notion of proper k-colorability.

Definition 1 *A* c**-coloring** *of a graph* G *with a vertex set* V *is a function from* V *to* $\{1, \ldots, c\}$*. Given a finite family (with repetitions)* $\mathcal{F}$ *of graphs, each provided with a* c*-coloring, an* $\mathcal{F}$**-coloring** *of* G *is a* c*-coloring* $\mathcal{C}$ *of* G *for which no member of* $\mathcal{F}$ *appears as an induced subgraph with the colors assigned to its vertices being identical to those assigned by* $\mathcal{C}$*.* G *is called* $\mathcal{F}$**-colorable** *if it admits an* $\mathcal{F}$*-coloring.*

This class of properties, which captures the essence (with regards to property testing) of the first order graph properties of type "$\exists\forall$", was shown in [2] to be testable, and a technique for using the Regularity Lemma of Szemerédi for obtaining results in the context of induced subgraphs was developed for this purpose.

While Theorem 1.1 seems tight from the point of view of first order expressions, many important testable graph properties are not expressible as instances of the first order expressions which were proven there to be testable. If the point of view of colorability is considered, further steps towards the understanding of the testable graph properties can be taken by considering further generalizations of the colorability notion defined above.

In the following, graph properties belonging to more general classes are shown to be testable, by combining some of the general methods of [2] with additional methods and arguments. It is also explained in the following how the results regarding testability (positive and negative) can be generalized to some other graph-related combinatorial structures.

Section 2 presents some definitions and lemmas regarding regular pairs, including the Regularity Lemma of Szemerédi, which are relevant to the proofs here, and Section 3 presents some of the general methods underlying these proofs. Section 4 shows the testability of graph properties belonging to the following class, which combines colorability with counting.

Definition 2 *Given a finite family* $\mathcal{F}$ *of* c*-colored graphs and a sequence* $\alpha = (\alpha_1, \ldots, \alpha_c)$ *of numbers satisfying* $0 \leq \alpha_i \leq 1$ *and* $\sum_{i=1}^{c} \alpha_i = 1$*, we call a graph* G *with* n *vertices* $(\alpha, \mathcal{F})$**-colorable** *if* G *admits an* $\mathcal{F}$*-coloring with the additional property that for every* i *the number of vertices of* G *which are colored with* i *is in* $\{\lfloor \alpha_i n \rfloor, \lceil \alpha_i n \rceil\}$*.*

This testability of $(\alpha, \mathcal{F})$-colorability (for every fixed, given, α and $\mathcal{F}$) implies the testability of some graph properties which are not expressible by colorability notions alone. An example of such

a property is that of a graph G with n vertices having a clique with $\lfloor \frac{1}{2}n \rfloor$ vertices (this particular property as well as some other properties involving vertex or edge count were actually proven already in [12] to be testable; the idea here is to find a common generalization of as many previous results as possible).

Since, unlike $\mathcal{F}$-colorability, $(\alpha, \mathcal{F})$-colorability is not a property which is inherited by induced subgraphs, special arguments which apply directly to most subgraphs of a given graph have to be used in conjunction with methods similar to those of [2].

One could argue that $(\alpha, \mathcal{F})$-colorability is not a generalization of $\mathcal{F}$-colorability, because in this property the ratio of the number of vertices colored with each color *must* be specified. We note here that with the methods of Section 4 it is also possible to show the testability of a property in which the ratio of the number of vertices colored with each color is in a specified range, rather than being up to rounding at a specified point. We concentrate on the latter case because proving the more general one would add verbiage without contributing additional insights.

Section 5 introduces some more definitions and lemmas, mostly dealing with a certain combinatorial structure related to graphs. These are used in Section 6 to prove the testability of all graph properties belonging to the following class.

Definition 3 *A* c**-pair-coloring** *of a graph* G *with a vertex set* V *is a function from* $[V]^2$*, the set of all (unordered) pairs of vertices of* G*, to* $\{1, \ldots, c\}$*. Given a finite family (with repetitions)* $\mathcal{F}$ *of* c*-pair-colored graphs, an* $\mathcal{F}$**-pair-coloring** *of* G *is a* c*-pair-coloring* $\mathcal{C}$ *of* G *for which no member of* $\mathcal{F}$ *appears as an induced subgraph with the colors assigned to its vertex pairs being identical to those assigned by* $\mathcal{C}$*.* G *is called* $\mathcal{F}$**-pair-colorable** *if it admits an* $\mathcal{F}$*-pair-coloring.*

This is clearly a generalization of the usual notion of edge-colorability, but every property expressible in terms of the above defined (vertex) $\mathcal{F}$-colorability is expressible in terms of this new notion as well. A vertex c-coloring is transformed into a $\binom{c+1}{2}$-pair-coloring by coloring each pair with the union of the colors assigned to its vertex members; it is not hard to ensure in this context (by forbidding a constant number of pair-colored graphs) that the allowable pair-colorings have corresponding vertex c-colorings.

Although the result of Section 6 applies to graphs, its proof requires additional arguments pertaining to the combinatorial structure defined in Section 5.

Finally, Section 7 summarizes some of what is known and what is not yet known regarding testing of graphs and related structures.

It should be noted that the dependence of the constants given here on ϵ is impractical, as a result of making use of the Regularity Lemma. This could be unavoidable, as there is presently no known testability proof even for the property of being triangle free, but the one making use of this lemma.

2 Regular pairs and equipartitions

For every two nonempty disjoint vertex sets A and B of a graph G, we define $e(A,B)$ to be the number of edges of G between A and B. The *edge density* of the pair is defined by $d(A,B) = \frac{e(A,B)}{|A||B|}$. We say that the pair A, B is *γ-regular*, if for any two subsets A' of A and B' of B, satisfying $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$, their edge density satisfies $|d(A',B') - d(A,B)| < \gamma$.

One simple yet useful property of regularity is that it is somewhat preserved when moving to subsets, as the following trivial lemma shows.

Lemma 2.1 *If A, B is a γ-regular pair with density δ, and $A' \subset A$ and $B' \subset B$ satisfy $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$ for some $\epsilon \geq \gamma$, then A', B' is a $\gamma \cdot \max\{2, \epsilon^{-1}\}$-regular pair with density at least $\delta - \gamma$ and at most $\delta + \gamma$.* $\square$

The following lemma shows how the existence of regular pairs implies the existence of many induced subgraphs of a fixed type. Many similar lemmas have been proven in previous works; for a complete proof of this one see [2].

Lemma 2.2 *For every $0 < \eta < 1$ and k there exist $\gamma = \gamma_{2.2}(\eta, k)$ and $\delta = \delta_{2.2}(\eta, k)$ with the following property.*

Suppose that H is a graph with vertices $v_1, \ldots, v_k$, and that $V_1, \ldots, V_k$ is a k-tuple of disjoint vertex sets of G such that for every $1 \leq i < i' \leq k$ the pair $V_i, V_{i'}$ is γ-regular, with density at least η if $v_i v_{i'}$ is an edge of H, and with density at most $1 - \eta$ if $v_i v_{i'}$ is not an edge of H. Then, at least $\delta \prod_{i=1}^{k} |V_i|$ of the k-tuples $w_1 \in V_1, \ldots, w_k \in V_k$ span (induced) copies of H where each w_i plays the role of v_i.

$\gamma_{2.2}(\eta, k)$ and $\delta_{2.2}(\eta, k)$ may and are assumed to be monotone nondecreasing in η and monotone nonincreasing in k. We also assume similar monotonicity properties for the other functions appearing in following lemmas.

We need to use the fact that most subpairs of a given regular pair are also regular to some extent. In order to prove such a lemma, we use the following alternative definition of regularity.

Definition 4 *For a pair U, W of vertex sets with density η, a vertex $u \in U$ is said to* **ϵ-agree** *with the pair if it has between $(\eta - \epsilon)|W|$ and $(\eta + \epsilon)|W|$ neighbors in W, and in addition all but at most $\epsilon|U|$ of the vertices $u' \in |U|$ are such that the number of mutual neighbors of u and u' in W is between $(\eta^2 - \epsilon)|W|$ and $(\eta^2 + \epsilon)|W|$. We define in an analogous manner the instance where a vertex $w \in W$ is said to ϵ-agree with the pair.*

The pair U, W is called **ϵ-locally-regular** *if all but at most $\epsilon|U|$ of the vertices of U and all but at most $\epsilon|W|$ of the vertices of W do ϵ-agree with the pair.*

It was proven in [1] that this notion of regularity is equivalent in a sense to the usual notion of regularity.

Lemma 2.3 ([1]) *For every $\epsilon > 0$ there exists $\gamma = \gamma_{2.3}(\epsilon) > 0$ such that every γ-locally-regular pair is also ϵ-regular, and every γ-regular pair is also ϵ-locally-regular.*

The following corollary shows that the regularity property is inherited by most subpairs, with any given size larger than a constant, of a given regular pair. It follows directly from lemma 2.3 by combining it with large deviation inequalities (see e.g. [4], Appendix A).

Corollary 2.4 *For every ϵ there exists $\gamma = \gamma_{2.4}(\epsilon)$, such that for every δ there exists $s = s_{2.4}(\epsilon, \delta)$ satisfying the following. If U, W is a γ-regular pair with density η, then for every $r > s$ and $t > s$, at least $1 - \delta$ of the possible choices of $U' \subset U$ and $W' \subset W$ with $|U'| = r$ and $|W'| = t$ satisfy that U', W' is an ϵ-regular pair whose density is between $\eta - \delta$ and $\eta + \delta$.*

Proof sketch: We set $\gamma = \gamma_{2.3}(\frac{1}{2}\gamma_{2.3}(\epsilon))$, and set s large enough to ensure that if U, W is a $\frac{1}{2}\gamma_{2.3}(\epsilon)$-locally-regular pair with density η, then for all but at most δ of the possible choices of U' and W' as above the pair U', W' is $\gamma_{2.3}(\epsilon)$-locally-regular and with density between $\eta - \delta$ and $\eta + \delta$. $\square$

A partition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ of the vertex set of a graph is called an *equipartition* if $|V_i|$ and $|V_{i'}|$ differ by no more than 1 for all $1 \leq i < i' \leq k$ (so in particular each V_i has one of two possible sizes). The order of such an equipartition $\mathcal{A}$ is the number of its sets (k in the above notation). A *refinement* of such an equipartition $\mathcal{A}$ is an equipartition of the form $\mathcal{B} = \{V_{i,j} | 1 \leq i \leq k, 1 \leq j \leq l\}$ such that $V_{i,j}$ is a subset of V_i for every $1 \leq i \leq k$ and $1 \leq j \leq l$.

In view of the above properties of regular pairs, it is useful to our purpose to ensure the existence of regular pairs in a way that nearly classifies a given input graph G. This is best done by ensuring the existence of a certain equipartition, using the Regularity Lemma of Szemerédi [20].

Lemma 2.5 (The Regularity Lemma [20]) *For every m and $\epsilon > 0$ there exists $T = T_{2.5}(m, \epsilon)$ with the following property.*

If G is a graph with $n \geq T$ vertices, and $\mathcal{A}$ is an equipartition of the vertex set of G with an order not exceeding m, then there exists a refinement $\mathcal{B}$ of $\mathcal{A}$ with order k, where $m \leq k \leq T$, for which all pairs of sets but at most $\epsilon\binom{k}{2}$ of them are ϵ-regular.

In the following, we assume that the number of vertices of the graph G, which appears in some of the formulations, is large enough (as a function of the other parameters) even when this is not mentioned explicitly.

The following corollary, many versions of which appear in various papers applying the Regularity Lemma, is useful in what follows. It is proven by combining Lemma 2.5 with Turán's Theorem and Ramsey's Theorem (see [5] for their formulation). See [2] for the full details.

Corollary 2.6 *For every h and γ there exists $\delta = \delta_{2.6}(h, \gamma)$ such that for every graph G with $n \geq \delta^{-1}$ vertices there exist disjoint vertex sets $W_1, \ldots, W_h$ satisfying:*

- *$|W_i| \geq \delta n$.*
- *All $\binom{h}{2}$ pairs are γ-regular.*
- *Either all pairs are with densities at least $\frac{1}{2}$, or all pairs are with densities less than $\frac{1}{2}$.*

In order to deal with non-monotone graph properties as well as monotone ones, the following variant of Lemma 2.5 from [2], which is suited for finding certain induced subgraphs in G, is used.

Lemma 2.7 ([2]) *For every integer m and function $0 < \mathcal{E}(r) < 1$ there exists $S = S_{2.7}(m, \mathcal{E})$ with the following property.*

If G is a graph with $n \geq S$ vertices, then there exists an equipartition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ and a refinement $\mathcal{B} = \{V_{i,j} | 1 \leq i \leq k, 1 \leq j \leq l\}$ of $\mathcal{A}$ that satisfy:

- *$|\mathcal{A}| = k \geq m$ but $|\mathcal{B}| = kl \leq S$.*
- *For all $1 \leq i < i' \leq k$ but at most $\mathcal{E}(0)\binom{k}{2}$ of them the pair $V_i, V_{i'}$ is $\mathcal{E}(0)$-regular.*
- *For all $1 \leq i < i' \leq k$, for all $1 \leq j, j' \leq l$ but at most $\mathcal{E}(k)l^2$ of them the pair $V_{i,j}, V_{i',j'}$ is $\mathcal{E}(k)$-regular.*
- *All $1 \leq i < i' \leq k$ but at most $\mathcal{E}(0)\binom{k}{2}$ of them are such that for all $1 \leq j, j' \leq l$ but at most $\mathcal{E}(0)l^2$ of them $|d(V_i, V_{i'}) - d(V_{i,j}, V_{i',j'})| < \mathcal{E}(0)$ holds.*

For many proofs regarding testability, including those in Section 6, but excluding those in Section 4, it is enough to use the following corollary.

Corollary 2.8 ([2]) *For every m and $0 < \mathcal{E}(r) < 1$ there exist $S = S_{2.8}(m, \mathcal{E})$ and $\delta = \delta_{2.8}(m, \mathcal{E})$ with the following property.*

If G is a graph with $n \geq S$ vertices then there exist an equipartition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ of G and an induced subgraph G' of G, with an equipartition $\mathcal{A}' = \{V'_i | 1 \leq i \leq k\}$ of the vertices of G', that satisfy:

- *$S \geq k \geq m$.*
- *$V'_i \subset V_i$ for all $i \geq 1$, and $|V'_i| \geq \delta n$.*
- *In the equipartition $\mathcal{A}'$,* **all** *pairs are $\mathcal{E}(k)$-regular.*
- *All but at most $\mathcal{E}(0)\binom{k}{2}$ of the pairs $1 \leq i < i' \leq k$ are such that $|d(V_i, V_{i'}) - d(V'_i, V'_{i'})| < \mathcal{E}(0)$.*

3 Small models for large graphs

Given a large graph with a partition of its vertex set, a possibly much smaller graph can capture many of its properties (at least with respect to worst cases) by having the regularity property. The following somewhat lengthy definition gives the properties required of this smaller graph.

Definition 5 *Given a graph $\tilde{G}$ and a partition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ of its vertices, a graph M is an* (h, s, t)**-based** (η, ϵ)**-model for** $\tilde{G}$ **over** $\mathcal{A}$ *if there is a partition $\mathcal{M} = \{W_{i,j} | 1 \leq i \leq k, 1 \leq j \leq h\}$ of its vertex set satisfying the following.*

- *$s \leq |W_{i,j}| \leq t$ for all $1 \leq i \leq k$ and $1 \leq j \leq h$.*
- *If $h > 1$, then for every fixed $1 \leq i \leq k$, all pairs $W_{i,j}, W_{i,j'}$ for $1 \leq j < j' \leq h$ are ϵ-regular. Moreover, if $h > 1$, then either all these pairs have densities at least $\frac{1}{2}$, in which case the subgraph of $\tilde{G}$ spanned by V_i is a clique, or all these pairs have densities less than $\frac{1}{2}$, in which case the subgraph of $\tilde{G}$ spanned by V_i is an edgeless graph.*
- *For every $1 \leq i < i' \leq k$ and $1 \leq j, j' \leq h$, the pair $W_{i,j}, W_{i',j'}$ is an ϵ-regular pair. Moreover, for every fixed $1 \leq i < i' \leq k$, one of the following three cases occurs. Either the densities of all the pairs $W_{i,j}, W_{i',j'}$ (where j and j' vary over $\{1, \ldots, h\}$) are between η and $1 - \eta$; or $\tilde{G}$*

contains all possible edges between V_i and $V_{i'}$, and the densities of all these pairs are at least η; or $\tilde{G}$ contains no edge between V_i and $V_{i'}$, and the densities of all these pairs are at most $1-\eta$.

In [2] it is proven implicitly that there exist certain induced subgraphs of G, which model a graph $\tilde{G}$ which has the same vertex set as G and an edge set similar to that of G. Let us review these proofs.

Existence in G of a fairly large induced subgraph, modeling a graph $\tilde{G}$ which is similar to G, follows readily from Corollary 2.8.

Lemma 3.1 (see [2]) *For every m, h and $0 < \mathcal{E}(r) < 1$ there exist $S = S_{3.1}(m,h,\mathcal{E})$ and $\delta = \delta_{3.1}(m,h,\mathcal{E})$, such that for every graph G there exists an equipartition $\mathcal{A} = \{V_i | 1 \le i \le k\}$ with $m \le k \le S$, a graph $\tilde{G}$ with the same vertex set as G which differs from G in less than $\mathcal{E}(0)\binom{n}{2}$ places, and an $(h,\delta n,n)$-based $(\frac{1}{6}\mathcal{E}(0),\mathcal{E}(k))$-model for $\tilde{G}$ over $\mathcal{A}$ which is an induced subgraph of G.*

Proof: We may assume that $\mathcal{E}(r)$ is monotone nonincreasing with r. For convenience, set $\epsilon = \mathcal{E}(0)$. We set

$$S = S_{2.8}(\max\{m, 7\epsilon^{-1}\}, \mathcal{E}')$$

using

$$\mathcal{E}'(r) = \min\{\frac{1}{6}\epsilon, \delta_{2.6}(h,\mathcal{E}(r))\mathcal{E}(r)\},$$

and set

$$\delta = \delta_{2.6}(h,\mathcal{E}(S))\delta_{2.8}(\max\{m, 7\epsilon^{-1}\}, \mathcal{E}').$$

We apply Corollary 2.8 to G, to find $\mathcal{A} = \{V_i | 1 \le i \le k\}$, G' and $\mathcal{A}' = \{V_i' | 1 \le i \le k\}$, that satisfy $\max\{m, 7\epsilon^{-1}\} \le k \le S_{2.8}(\max\{m, 7\epsilon^{-1}\}, \mathcal{E}')$ and $|V_i'| \ge \delta_{2.8}(\max\{m, 7\epsilon^{-1}\}, \mathcal{E}')n$, and the regularity and density properties guaranteed by the corollary with regards to $\mathcal{E}'(r)$.

We use Corollary 2.6 on the subgraph of G induced by each V_i' to obtain $W_{i,1}, \ldots, W_{i,h}$, with all pairs being $\mathcal{E}(k)$-regular, and either all of the densities being at least $\frac{1}{2}$ or all of them being less than $\frac{1}{2}$. Note that now all pairs $W_{i,j}, W_{i',j'}$ are $\mathcal{E}(k)$-regular, and that their densities do not differ from those of $V_i', V_{i'}'$ by more than $\frac{1}{6}\epsilon$.

$\tilde{G}$ is defined to be the graph obtained from G by adding and removing edges according to what follows.

- For $1 \le i < i' \le k$ such that $|d(V_i, V_{i'}) - d(V_i', V_{i'}')| > \frac{1}{6}\epsilon$, for all $v \in V_i$ and $v' \in V_{i'}$ the pair vv' becomes an edge if $d(V_i', V_{i'}') \ge \frac{1}{2}$, and becomes a non-edge otherwise. This changes less than $\frac{2}{6}\epsilon\binom{n}{2}$ edges (for n large enough) because there are no more than $\frac{1}{6}\epsilon\binom{k}{2}$ such $1 \le i < i' \le k$.

- For $1 \le i < i' \le k$ such that $d(V_i', V_{i'}') < \frac{2}{6}\epsilon$, all edges between V_i and $V_{i'}$ are removed. For all $1 \le i < i' \le k$ such that $d(V_i', V_{i'}') > 1 - \frac{2}{6}\epsilon$, all non-edges between V_i and $V_{i'}$ become edges. This changes no more than $\frac{3}{6}\epsilon\binom{n}{2}$ edges in addition to those changed by the previous operation.

- For $1 \le i < i' \le k$ such that none of the above holds, the edges of $\tilde{G}$ between V_i and $V_{i'}$ remain exactly those of G.

- If $h > 1$ we perform the following. If for a fixed i all densities of pairs from $W_{i,1}, \ldots, W_{i,h}$ are less than $\frac{1}{2}$, all edges within the vertices of V_i are removed. Otherwise, all the abovementioned densities are at least $\frac{1}{2}$, in which case all non-edges within V_i become edges. This changes less than $\frac{1}{6}\epsilon\binom{n}{2}$ edges, because $k \ge 7\epsilon^{-1}$.

G' is clearly a model graph for $\tilde{G}$ as in the formulation of this lemma, and according to the calculation above $\tilde{G}$ differs from G in less than $\epsilon\binom{n}{2}$ places, as required. □

By combining this with Lemma 2.2 the following corollary is immediate (a detailed proof of a similar one is implied in [2]).

Corollary 3.2 (see [2]) *For every m, h, $\mathcal{S}(r)$ and $0 < \mathcal{E}(r) < 1$ there exist $S = S_{3.2}(m, h, \mathcal{S}, \mathcal{E})$, $t = t_{3.2}(m, h, \mathcal{S}, \mathcal{E})$ and $\delta = \delta_{3.2}(m, h, \mathcal{S}, \mathcal{E})$, such that for every graph G with $n > S$ vertices there exists an equipartition $\mathcal{A} = \{V_i | 1 \le i \le k\}$ with $m \le k \le S$, and a graph $\tilde{G}$ with the same vertex set as G which differs from G in less than $\mathcal{E}(0)\binom{n}{2}$ places, such that at least $\delta\binom{n}{t}$ of the induced subgraphs of G with t vertices are $(h, \mathcal{S}(k), t)$-based $(\frac{1}{3}, \mathcal{E}(k))$-models for $\tilde{G}$ over $\mathcal{A}$.*

Proof sketch: We may assume that $\mathcal{S}(r)$ is monotone nondecreasing. We construct $\tilde{G}$ and $\mathcal{A}$, and then describe the required model graphs. Apply Lemma 3.1 (using appropriate m' and $\mathcal{E}'(r)$) to G, to obtain $\mathcal{A}$, $\tilde{G}$ and G' which satisfy the conditions in its formulation. We now consider any graph H with t vertices and an equipartition $\{W_{i,j}' | 1 \le i \le k, 1 \le j \le h\}$ which satisfies the following, where $t \ge (\mathcal{S}(S) + 1)S$ is chosen to be large enough for the construction below to be possible.

- For every $1 \le i \le k$ if $h > 1$, if the restriction of $\tilde{G}$ to V_i is a clique, then H contains all possible edges between $W_{i,j}'$ and $W_{i,j'}'$ for any $1 \le j < j' \le h$; otherwise the restriction of $\tilde{G}$ to V_i is edgeless, in which case H contains no edges between $W_{i,j}'$ and $W_{i,j'}'$.

- For every $1 \leq i < i' \leq k$, if $\tilde{G}$ contains all edges between V_i and $V_{i'}$, then H contains all edges between $W_{i,j}$ and $W_{i',j'}$ for any $1 \leq j, j' \leq h$; if $\tilde{G}$ contains no edges between V_i and $V_{i'}$, then H contains no edges between $W_{i,j}$ and $W_{i',j'}$; and if non of the above occurs, then $W_{i,j}, W_{i',j'}$ is some fixed $\mathcal{E}(k)$-regular pair whose density is between $\frac{1}{3}$ and $\frac{2}{3}$ (a fixed regular pair satisfying these requirements exists if t is chosen to make $\lfloor \frac{t}{k} \rfloor \geq \lfloor \frac{t}{S} \rfloor$ large enough).

It follows readily that any such H is a model graph for $\tilde{G}$, and Lemma 2.2 ensures that enough of the induced subgraphs of G' satisfy the above for the statement of the proposition to hold for an appropriately chosen δ. $\square$

The above corollary is sufficient for many proofs of testability of colorability properties, because it guarantees that most large enough random induced subgraphs of G contain an induced model graph for an appropriate $\tilde{G}$, and this model can in turn be shown to be colorable only if $\tilde{G}$ is. However, in the case where the number of appearances of each color is also important, a more refined mean of obtaining model graphs must be used.

To shorten notation, in the following we call a c-coloring an *α-coloring* if it is an $(\alpha, \mathcal{F})$-coloring for some $\mathcal{F}$, e.g. $\mathcal{F} = \emptyset$. We also use some vector notation regarding α. For example, $|\alpha - \alpha'|_\infty$ is the largest difference between a component of α and the corresponding component of α'.

Given a model graph M (for some $\tilde{G}$ over some partition $\mathcal{A}$ of its vertices), and a partition $\mathcal{M} = \{W_{i,j} | 1 \leq i \leq k, 1 \leq j \leq h\}$ of the vertices of M as in the notation of the above definition of model graphs, we say that a c-coloring of M is *α-modeling* if its restriction to $\bigcup_{j=1}^{h} W_{i,j}$ is monochromatic for every fixed $1 \leq i \leq k$, and, denoting the common color of the above set by c_i, satisfies also that $|\{i | c_i = g\}| = \alpha_g k$ for every fixed $1 \leq g \leq c$. We are now ready to formulate the lemma regarding existence of model graphs which is required for proving the testability of $(\alpha, \mathcal{F})$-colorability properties.

Lemma 3.3 *For every h, c and function $\mathcal{E}(r)$ there exist $S = S_{3.3}(h, c, \mathcal{E})$, $\delta = \delta_{3.3}(h, c, \mathcal{E})$ and $s = s_{3.3}(h, c, \mathcal{E})$, such that for every graph G with $n > s$ vertices there exists an equipartition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ with $k \leq S$, satisfying that for every $t \geq s$, at least $(1 - \mathcal{E}(0))\binom{n}{t}$ of the induced subgraphs G' of G with t vertices satisfy the following.*

For every α-coloring of G' (for any $\alpha = (\alpha_1, \ldots, \alpha_c)$ satisfying $\alpha_i \geq 0$ and $\sum_{i=1}^{c} \alpha_i = 1$), there exists an induced subgraph G'' of G', which is an $(h, \delta t, t)$-based $(\frac{1}{6}\mathcal{E}(0), \mathcal{E}(k))$-model over $\mathcal{A}$ for a graph $\tilde{G}$ which differs from G in less than $\mathcal{E}(0)\binom{n}{2}$ places, so that the restriction of the given coloring of G' to G'' is α'-modeling for some α' satisfying $|\alpha - \alpha'|_\infty \leq \mathcal{E}(0)$.

Proof: We may assume $\mathcal{E}(r) \leq \mathcal{E}(0)$, and set $\epsilon = \mathcal{E}(0)$. Define $\mathcal{E}'$ by

$$\mathcal{E}'(r) = \min\left\{\gamma_{2.4}\left(\min\{\frac{1}{4cr}\delta_{2.6}\left(h, \mathcal{E}(r)\right)\mathcal{E}(r), \frac{1}{6}\epsilon\}\right), \frac{1}{192}\epsilon, \frac{1}{16}\binom{r+2}{2}^{-1}\right\},$$

set

$$S = S_{2.7}(\max\{m, 7\epsilon^{-1}\}, \mathcal{E}')$$

with m to be chosen later, and set

$$\delta = \frac{1}{6cS^2}\delta_{2.6}(h, \mathcal{E}(S)).$$

Use Lemma 2.7 on G, to find the partitions $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ and $\mathcal{B} = \{V_{i,j} | 1 \leq i \leq k, 1 \leq j \leq l\}$ which satisfy the density and regularity properties in its formulation; the partition $\mathcal{A}$ will also play the role of the one appearing in the formulation here.

We choose

$$s = \max\left\{2Ss_{2.4}\left(\min\{\frac{1}{4cS}\delta_{2.6}\left(h, \mathcal{E}(S)\right)\mathcal{E}(S), \frac{1}{6}\epsilon\}, \min\{\frac{1}{12}\epsilon, \frac{1}{2}\epsilon\binom{S}{2}^{-1}\}\right), s_0\right\},$$

where s_0 is chosen so that with probability at least $1 - \frac{1}{2}\epsilon$ a random subgraph G' with $t \geq s$ vertices contains between $\frac{2t}{3kl}$ and $\frac{4t}{3kl}$ vertices from each $V_{i,j}$. Thus, denoting by $V'_{i,j}$ the set of vertices of G' belonging to $V_{i,j}$, by Corollary 2.4 the following properties hold in particular with probability at least $1 - \epsilon$ (these properties are similar to those of the equipartition $\mathcal{B}$).

- $\frac{2t}{3kl} \leq |V'_{i,j}| \leq \frac{4t}{3kl}$
- For all $1 \leq i < i' \leq k$, for all $1 \leq j, j' \leq l$ but at most $\frac{1}{16}\binom{k}{2}^{-1}l^2$ of them the pair $V'_{i,j}, V'_{i',j'}$ is $\min\{\frac{1}{4ck}\delta_{2.6}(h, \mathcal{E}(k))\mathcal{E}(k), \frac{1}{6}\epsilon\}$-regular.
- All $1 \leq i < i' \leq k$ but at most $\frac{1}{192}\epsilon\binom{k}{2} < \frac{1}{12}\epsilon\binom{k}{2}$ of them are such that for all $1 \leq j, j' \leq l$ but at most $\frac{1}{192}\epsilon l^2$ of them $|d(V_i, V_{i'}) - d(V'_{i,j}, V'_{i',j'})| \leq (\frac{1}{192} + \frac{1}{12})\epsilon < \frac{1}{6}\epsilon$ holds.

We now show that such a graph G' has the properties described in the formulation of this lemma. Given an α-coloring $\mathcal{C}$ of G', choose independently, uniformly, and randomly, a vertex $v_i \in \bigcup_{j=1}^{l} V'_{i,j}$ for every $1 \leq i \leq k$. Choose m to be large enough to ensure (by large deviation inequalities) that with probability more than $\frac{3}{4}$, the restriction of $\mathcal{C}$ to the subgraph induced by $v_1, \ldots, v_k$ is an α'-coloring for some α' satisfying $|\alpha - \alpha'|_\infty \leq \epsilon$ (this choice depends only on ϵ). Denoting by j_i the index for which $v_i \in V'_{i,j_i}$, with probability at least $\frac{3}{4}$ the number of vertices

in V'_{i,j_i} which have the same color as v_i is at least $\frac{1}{4ck}|V'_{i,j}|$ (because for every fixed $1 \leq i \leq k$ and $1 \leq g \leq c$, the probability of v_i to be colored g while the number of the other vertices of V'_{i,j_i} so colored is less than $\frac{1}{4ck}|V'_{i,j_i}|$, is bounded by $\frac{1}{4ck}$). Finally, because for every fixed i and j the probability that $j_i = j$ is no more than $\frac{2}{l}$, each of the following properties holds with probability at least $\frac{3}{4}$.

- Within the family $\{V'_{1,j_1}, \ldots, V'_{k,j_k}\}$, all pairs are $\min\{\frac{1}{4ck}\delta_{2.6}(h, \mathcal{E}(k))\mathcal{E}(k), \frac{1}{6}\epsilon\}$-regular; this happens with probability at least $\frac{3}{4}$ because for every fixed $1 \leq i < i' \leq k$ the probability of the pair $V'_{i,j_i}, V'_{i',j_{i'}}$ to be otherwise is no more than $\frac{1}{4}\binom{k}{2}^{-1}$.
- All $1 \leq i < i' \leq k$ but at most $\frac{1}{6}\epsilon\binom{k}{2}$ of them satisfy $|d(V'_{i,j_i}, V'_{i',j_{i'}}) - d(V_i, V_{i'})| < \frac{1}{6}\epsilon$. This happens with probability at least $\frac{3}{4}$ because the average number of pairs $1 \leq i < i' \leq k$ that do not satisfy $|d(V'_{i,j_i}, V'_{i',j_{i'}}) - d(V_i, V_{i'})| < \frac{1}{6}\epsilon$, but do satisfy $|d(V_i, V_{i'}) - d(V'_{i,j}, V'_{i',j'})| < \frac{1}{6}\epsilon$ for all but at most $\frac{1}{192}\epsilon l^2$ of the possible $1 \leq j, j' \leq l$, is no more than $\frac{1}{48}\epsilon\binom{k}{2}$; thus with probability at least $\frac{3}{4}$ there are no more than $\frac{1}{12}\epsilon\binom{k}{2}$ such $1 \leq i < i' \leq k$.

In particular, there exists a choice of $v_1, \ldots, v_k$, with the appropriate $j_1, \ldots, j_k$, for which all of the above properties hold. We fix one such choice. Now, for every $1 \leq i \leq k$, we apply Corollary 2.6 to the subgraph induced by the set of all vertices in V'_{i,j_i} which are colored by $\mathcal{C}$ with the same color as v_i, thus obtaining sets $W_{i,1}, \ldots, W_{i,h}$ such that all their members share the same color, all pairs are $\mathcal{E}(k)$-regular, and their densities are either all at least $\frac{1}{2}$ or all less than $\frac{1}{2}$. Note that also all the pairs $W_{i,j}, W_{i',j'}$ are $\mathcal{E}(k)$-regular.

The subgraph G'' is defined as that spanned by the union of $\{W_{i,j} | 1 \leq i \leq k, 1 \leq j \leq h\}$. By constructing $\tilde{G}$ now in a manner completely analogous to the construction in the proof of Lemma 3.1, G'' is proven to be the required subgraph. $\square$

4 Vertex-colorability with restrictions

This section contains a proof that graph properties defined in terms of (vertex) $(\alpha, \mathcal{F})$-colorability are testable. For convenience, an h-based (η, ϵ)-model graph refers to an (h, s, t)-based (η, ϵ)-model graph for any $1 \leq s \leq t$, e.g. $s = 1$ and $t = \infty$ (the parameters s and t play a lesser role in this particular section). The main combinatorial result follows from the following lemma about the impossibility that a graph far from being colorable has a model graph with a modeling coloring, by combining it with Lemma 3.3 about the existence of such a setting.

Lemma 4.1 *For every $\epsilon > 0$ and $\alpha = (\alpha_1, \ldots, \alpha_c)$ satisfying $0 \le \alpha_i \le 1$ and $\sum_{i=1}^{c} \alpha_i = 1$ there exist $\beta = \beta_{4.1}(\epsilon, \alpha)$, such that for every $\eta > 0$ and family $\mathcal{F}$ of c-colored graphs with up to h vertices in each member, there exists $\gamma = \gamma_{4.1}(\eta, \mathcal{F})$ satisfying the following.*

If an h-based (η, γ)-model for $\tilde{G}$ over $\mathcal{A} = \{V_i | 1 \le i \le k\}$ has an α'-modeling $\mathcal{F}$-coloring for any α' satisfying $|\alpha - \alpha'|_\infty < \beta$, then there exists an $(\alpha, \mathcal{F})$-colorable graph $\hat{G}$ which differs from $\tilde{G}$ in no more than $\epsilon\binom{n}{2}$ places, where n is the number of vertices of $\tilde{G}$.

Proof: Without loss of generality, assume that α_1 is the smallest non-zero coordinate of α. We choose

$$\beta = \min\{\frac{1}{2}\alpha_1, \frac{1}{2c}\epsilon\}$$

and

$$\gamma = \gamma_{2.2}(\eta, h).$$

Given $\tilde{G}$, the model graph H with its partition $\{W_{i,j} | 1 \le i \le k, 1 \le j \le h\}$, and an α'-modeling $\mathcal{F}$-coloring $\mathcal{C}$ of H satisfying the above, we generate $\hat{G}$, its partition $\hat{\mathcal{A}} = \{\hat{V}_i | 1 \le i \le k\}$ and its $(\alpha, \mathcal{F})$-coloring $\mathcal{D}$ as follows.

Since $|\alpha - \alpha'|_\infty < \alpha_1$, for every $1 \le g \le c$ either $\alpha_g = 0$ or $\alpha'_g > 0$. In the latter case, let i_g denote one index such that the vertices of $\bigcup_{j=1}^{h} W_{i_g,j}$ are all colored g. Now color as many vertices of $\tilde{G}$ as possible, under the following restrictions. For every i, every vertex of V_i is either colored the same as $\bigcup_{j=1}^{h} W_{i,j}$, or remains uncolored; and for every $1 \le g \le c$, no more than $\lfloor \alpha n \rfloor$ vertices of $\tilde{G}$ are colored g. In the following construction of $\hat{\mathcal{A}}$, for every i the set $\hat{V}_i$ holds the vertices of V_i which are colored in this stage, and may hold other vertices as follows. Since $|\alpha - \alpha'|_\infty \le \frac{1}{2c}\epsilon$, no more than $\frac{1}{2}\epsilon n$ vertices remain uncolored. To complete the construction of the coloring $\mathcal{D}$, for every uncolored vertex v, a color $1 \le g \le c$ is chosen such that the final coloring $\mathcal{D}$ is an α-coloring; to complete the construction of $\hat{\mathcal{A}}$ and $\hat{G}$, such a v is placed in $\hat{V}_{i_g}$, and all its incident edges are modified so that H is also a model for $\hat{G}$ over $\hat{\mathcal{A}}$.

It remains to prove that $\mathcal{D}$ is an $\mathcal{F}$-coloring. Supposing otherwise, assume that $v_1, \ldots, v_h$ span a graph which appears in $\mathcal{F}$ with the same colors as $\mathcal{D}$. For every $1 \le j \le h$, let $\hat{\imath}_j$ denote the index for which $v_j \in \hat{V}_{\hat{\imath}_j}$. Now, considering the sets $W_{\hat{\imath}_1,1}, \ldots, W_{\hat{\imath}_h,h}$, Lemma 2.2 ensures the existence of $w_1 \in W_{\hat{\imath}_1,1}, \ldots, w_h \in W_{\hat{\imath}_h,h}$ that span a graph which appears in $\mathcal{F}$ with the same colors as $\mathcal{C}$, but this is a contradiction as $\mathcal{C}$ is an $\mathcal{F}$-coloring of H. $\square$

Theorem 4.2 *For every ϵ, $\mathcal{F}$, and $\alpha = (\alpha_1, \ldots, \alpha_c)$ satisfying $0 \le \alpha_i \le 1$ and $\sum_{i=1}^{c} \alpha_i = 1$, there exist $s = s(\epsilon, \alpha, \mathcal{F})$ and $\beta = \beta(\epsilon, \alpha, \mathcal{F})$, such that for every graph G with n vertices which is ϵ-far*

from being $(\alpha, \mathcal{F})$-colorable, and for every $t \geq s$, at least $(1-\epsilon)\binom{n}{t}$ of the sets with t vertices of G span graphs which are not $(\alpha', \mathcal{F})$-colorable for any α' satisfying $|\alpha - \alpha'|_\infty < \beta$.

Proof: We set $\beta = \frac{1}{2}\beta_{4.1}(\frac{1}{2}\epsilon, \alpha)$. We define $\mathcal{E}$, by setting

$$\mathcal{E}(0) = \min\{\frac{1}{2}\epsilon, \beta\},$$

and for every $r > 0$ setting

$$\mathcal{E}(r) = \gamma = \gamma_{4.1}(\frac{1}{6}\mathcal{E}(0), \mathcal{F}).$$

Finally, we set $s = s_{3.3}(h, c, \mathcal{E})$, where c is the number of colors, and h is the maximal number of vertices in a member of $\mathcal{F}$.

If G is ϵ-far from being $(\alpha, \mathcal{F})$-colorable, then more than $(1-\epsilon)\binom{n}{t}$ of the induced subgraphs of G with t vertices (for any fixed $t \geq s$) have the property given in the formulation of Lemma 3.3. If any of these is $(\alpha', \mathcal{F})$-colorable for some α' satisfying $|\alpha - \alpha'|_\infty < \beta$, then it contains a $(\frac{1}{6}\mathcal{E}(0), \gamma)$-model graph for a graph $\tilde{G}$ which differs from G in less than $\frac{1}{2}\epsilon\binom{n}{2}$ places, with an α''-modeling $\mathcal{F}$-coloring for some α'' which satisfies $|\alpha - \alpha''|_\infty < 2\beta$. But then Lemma 4.1 ensures the existence of $\hat{G}$ which differs from G in less than $\epsilon\binom{n}{2}$ places and is also $(\alpha, \mathcal{F})$-colorable, which is a contradiction. $\square$

Corollary 4.3 *For every fixed α and $\mathcal{F}$, the property of a graph being $(\alpha, \mathcal{F})$-colorable is testable.*

Proof: Given ϵ, an ϵ-test for this property is constructed as follows. Let $\beta = \beta(\min\{\frac{1}{3}, \epsilon\}, \alpha, \mathcal{F})$ be as in Theorem 4.2, and let $t \geq s(\min\{\frac{1}{3}, \epsilon\}, \alpha, \mathcal{F})$ be large enough to ensure also that given an α-coloring of any graph G with n vertices, at least $\frac{2}{3}\binom{n}{t}$ of its restrictions to sets of t vertices of G are α'-colorings for some α' satisfying $|\alpha - \alpha'|_\infty < \beta$ (such a t independent of n can be found using large deviation inequalities).

The test is as follows: A uniformly random set of t vertices of G is chosen, and all pairs of this sets are queried. The algorithm accepts G if and only if the subgraph of G induced by this set is $(\alpha', \mathcal{F})$-colorable for any α' satisfying $|\alpha - \alpha'|_\infty < \beta$ (this can be checked by an exhaustive search of all possible c-colorings of this subgraph). If G is $(\alpha, \mathcal{F})$-colorable, the existence of such a coloring ensures, by the choice of t, that with probability at least $\frac{2}{3}$ the algorithm accepts. If G is ϵ-far from being $(\alpha, \mathcal{F})$-colorable, Theorem 4.2 ensures that with probability at least $\frac{2}{3}$ the algorithm rejects. $\square$

5 Regularity and charts

This section deals with a combinatorial structure which is a generalization of a graph. It turns out that this structure is useful for proving testability of graph properties which are defined in terms of edge or pair colorings. In the following, recall that $[V]^2$ denotes the set of the unordered pairs of members of V.

Definition 6 *A* g**-chart** *is an ordered pair* $C = (V, P)$ *consisting of a set of* **vertices** V *and a function* $P : [V]^2 \to \{1 \dots g\}$*, which is called the* **pair function** *of* C*.*

In particular, a graph may be regarded as an instance of a 2-chart, and a c-pair-colored graph may be regarded as an instance of a $2c$-chart. The notion of a subchart induced by a subset of the vertex set of the chart is defined in an analogous manner to the notion of an induced subgraph. The *preimage graphs* $C_1, \dots, C_g$ of a g-chart C are defined as the graphs having the same vertex set as C, with the edges of each C_j being exactly the pairs of C for which its pair function gives the value j.

For $1 \le j \le g$ we define the j*-density* of a pair A, B of vertex sets of a g-chart C as its density when considered as a pair of vertex sets of the preimage graph C_j of C. We call such a pair γ*-regular* if it is γ-regular considered as a pair of vertex sets of each of the preimage graphs $C_1, \dots, C_g$ of C. Most of the results related to regularity in graphs have counterparts for charts. Many similar lemmas have been proven in previous works, but for completeness we give some of the proofs here.

Lemma 5.1 *If* A, B *is a* γ*-regular pair in a chart, and* $A' \subset A$ *and* $B' \subset B$ *satisfy* $|A'| \ge \epsilon|A|$ *and* $|B'| \ge \epsilon|B|$ *for some* $\epsilon \ge \gamma$*, then* A', B' *is a* $\gamma \cdot \max\{2, \epsilon^{-1}\}$*-regular pair.*

Proof: We consider the preimage graphs $C_1, \dots, C_g$ of our chart C. The γ-regularity of A, B implies that it is also a γ-regular pair with respect to each of the preimage graphs, and so by Lemma 2.1 the pair A', B' is $\gamma \cdot \max\{2, \epsilon^{-1}\}$-regular with respect of each of $C_1, \dots, C_g$. But this means that it is also $\gamma \cdot \max\{2, \epsilon^{-1}\}$-regular with respect to the chart C. $\square$

Lemma 5.2 (Chart version of Lemma 2.2) *For every* $\eta > 0$ *and* h *there exist* $\gamma = \gamma_{5.2}(\eta, h)$ *and* $\delta = \delta_{5.2}(\eta, h)$ *with the following property.*

Suppose that C *and* $D = (\{v_1, \dots, v_h\}, P)$ *are charts, and that* $V_1, \dots, V_h$ *are disjoint vertex sets of* C *such that for every* $1 \le i < i' \le h$ *the pair* $V_i, V_{i'}$ *is* γ*-regular and its* $P(v_i v_{i'})$*-density is at least* η*. Then, at least* $\delta \prod_{i=1}^{h} |V_i|$ *of the* h*-tuples* $w_1 \in V_1, \dots, w_h \in V_h$ *span copies of* D *where each* w_i *plays the role of* v_i*.*

Proof: We construct a graph with the vertex set $V_1, \ldots, V_h$ as follows: There are no edges between any w and w' that belong to the same V_i; for every $i < i'$ there is an edge between $w \in V_i$ and $w' \in V_{i'}$ if and only if the pair function of C assigns to w, w' the value $P(v_i, v_{i'})$.

We now note that if all the pairs were $\gamma_{2.2}(h, \eta)$ regular in the original chart, then the constructed graph satisfies the conditions of Lemma 2.2 for the existence of $\delta_{2.2} \prod_{i=1}^{h} |V_i|$ copies of the clique with h vertices; each such copy corresponds to an h-tuple $w_1 \in V_1, \ldots, w_h \in V_h$ as in the assertion of our lemma, concluding the proof. $\square$.

Lemma 5.3 (Regularity Lemma for charts) *For every g, m and $\epsilon > 0$ there exists a number $T = T_{5.3}(g, m, \epsilon)$ with the following property.*

If C is a g-chart with $n \geq T$ vertices, and $\mathcal{A}$ is an equipartition of the vertex set of C with an order not exceeding m, then there exists a refinement $\mathcal{B}$ of $\mathcal{A}$ of order k, where $m \leq k \leq T$, for which all pairs of sets but at most $\epsilon\binom{k}{2}$ of them are ϵ-regular.

Proof sketch: The proof of the Regularity Lemma on graphs depends on the construction of an index function for vertex partitions of the graph. The proof then proceeds by a lemma that states that if a partition is not ϵ-regular, then there exists a refinement of this partition, whose order is a function of the order of the original partition, and whose index is higher by a constant amount (depending on ϵ). As the index of a partition is always between 0 and 1, the proof follows.

For the case of charts we look for every $1 \leq j \leq g$ at the index of the partition with respect of C_j. We can then define the index of the partition with respect to the chart to be the sum of these functions (its value is always between 0 and g), and then repeat almost word for word the proof that is used for the case of graphs. $\square$

By using this in conjunction with Turán's Theorem and a chart version of Ramsey's Theorem, we arrive at the following chart version of Corollary 2.6.

Corollary 5.4 (Chart version of Corollary 2.6) *For every g, h and γ there exists a quantity $\delta = \delta_{5.4}(g, h, \gamma)$ such that for every g-chart C with $n \geq \delta^{-1}$ vertices there exist disjoint vertex sets $W_1, \ldots, W_h$ satisfying:*

- $|W_i| \geq \delta n$.
- *All $\binom{h}{2}$ pairs are γ-regular.*
- *There exists some $1 \leq j \leq g$ such that the j-density of every pair is at least $\frac{1}{g}$.*

Proof sketch: We let r be such that any g-chart with r vertices has a subchart with h vertices whose pair function is constant (the existence of such an r is a consequence of Ramsey's Theorem). By use of the chart version of the Regularity Lemma in conjunction with Turán's Theorem (see e.g. the proof that appears in [2] for the above Corollary 2.6) we find in C vertex subsets $V_1, \ldots, V_r$ such that $|V_j| \geq \delta n$ for every j and for which all pairs are γ-regular.

We now construct a chart $D = (v_1, \ldots, v_r, P)$ for which P satisfies that the $P(v_k, v_{k'})$-density of $V_k, V_{k'}$ in C is at least $\frac{1}{g}$ for every $k < k'$. Denoting by $v_{k_1}, \ldots, v_{k_h}$ the subchart of D whose pair function is constant, and letting j (as in the formulation of the corollary) be this constant, we define $W_i = V_{k_i}$ for every $1 \leq i \leq h$. $\square$

We also need the following simple lemma about charts (it is simpler than Corollary 2.8 because nothing is claimed about the densities of the regular pairs it obtains).

Lemma 5.5 *For every g, k and γ there exists $\delta = \delta_{5.5}(g, k, \gamma)$ such that for every g-chart C with $n > \delta^{-1}$ vertices, and every equipartition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ of its vertices, there exist $V_i' \subset V_i$ such that $|V_i'| \geq \delta n$ for every $1 \leq i \leq k$ and $V_i', V_{i'}'$ is γ-regular for every $1 \leq i < i' \leq k$.*

Proof: We set $\delta = \frac{1}{2}(T_{5.3}(g, k, \min\{\gamma, \frac{1}{2}\binom{k}{2}^{-1}\}))^{-1}$. We apply Lemma 5.3 to find a refinement $\mathcal{B} = \{V_{i,j} | 1 \leq i \leq k, 1 \leq j \leq l\}$ of $\mathcal{A}$ with all but at most $\frac{1}{2}\binom{k}{2}^{-1}\binom{kl}{2}$ of its pairs being γ-regular.

Choose randomly, independently and uniformly $1 \leq j_i \leq l$ for each $1 \leq i \leq k$. Clearly, with probability at least $\frac{1}{2}$, all the pairs $V_{i,j_i}, V_{i',j_{i'}}$ are γ-regular. Fixing one such choice of $j_1, \ldots, j_k$, we set $V_i' = V_{i,j_i}$ to obtain the required result. $\square$

6 Pair-colorability

In this section we prove that graph properties defined in terms of $\mathcal{F}$-pair-colorability are testable. The main combinatorial result follows from the following lemma about the possible colorability of a model graph, by combining it with Corollary 3.2 about the existence of such model graphs.

Lemma 6.1 *For every k and family $\mathcal{F}$ of c-pair-colored graphs there exist $\gamma = \gamma_{6.1}(k, \mathcal{F})$ and $s = s_{6.1}(k, \mathcal{F})$, such that if a $(1, s, n)$-based $(\frac{1}{3}, \epsilon)$-model graph M for a graph $\tilde{G}$ over a partition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ is $\mathcal{F}$-pair-colorable, then there exists a graph $\hat{G}$ which may differ from $\tilde{G}$ only on the pairs $\bigcup_{i=1}^{k} [V_i]^2$ and is $\mathcal{F}$-pair-colorable.*

Proof: Let h denote the maximum number of vertices in a member of $\mathcal{F}$. We choose

$$\gamma = \min\left\{\frac{1}{9}, \delta_{5.5}\left(2c, k, \min\{\frac{1}{9c}, \delta_{5.4}\left(2c, h, \gamma_{5.2}(\frac{1}{9c}, h)\right)\gamma_{5.2}(\frac{1}{9c}, h)\}\right)\right\}$$

and

$$s = (\delta_{5.5}\left(2c, k, \min\{\frac{1}{9c}, \delta_{5.4}\left(2c, h, \gamma_{5.2}(\frac{1}{9c}, h)\right)\gamma_{5.2}(\frac{1}{9c}, h)\}\right)\delta_{5.4}\left(2c, h, \gamma_{5.2}(\frac{1}{9c}, h)\right))^{-1}.$$

Assume that $\{U_i | 1 \leq i \leq k\}$ is the partition of M corresponding to the partition $\mathcal{A}$ of $\tilde{G}$ (so U_i plays the role of $W_{1,i}$ in the notation of the definition of model graphs). Assuming that $\mathcal{C}$ is an $\mathcal{F}$-pair-coloring of M, we show how to construct $\hat{G}$ and an $\mathcal{F}$-pair-coloring $\mathcal{D}$ of $\hat{G}$. We consider M with its coloring $\mathcal{C}$ as a $2c$-chart, and use Lemma 5.5 to find $U'_i \subset U_i$ such that all pairs of $\mathcal{A}' = \{U'_i | 1 \leq i \leq k\}$ are $\min\{\frac{1}{9c}, \delta_{5.4}(2c, h, \gamma_{5.2}(\frac{1}{9c}, h))\gamma_{5.2}(\frac{1}{9c}, h)\}$-regular as pairs of a $2c$-chart. To each U'_i we apply Corollary 5.4 to obtain $\{U'_{i,j} | 1 \leq i \leq k, 1 \leq j \leq h\}$, with all pairs being $\gamma_{5.2}(\frac{1}{9c}, h)$-regular. $\hat{G}$ is constructed from $\tilde{G}$ as follows.

For every fixed i, if there is a color such that the density of the edges colored with this color is at least $\frac{1}{2c} > \frac{1}{9c}$ in all the pairs of $\{U'_{i,j} | 1 \leq j \leq h\}$, we denote it by k_i and add to $\hat{G}$ the all possible edges in $[V_i]^2$. Otherwise there exists a color such that the density of the non-edges colored with this color is at least $\frac{1}{2c} > \frac{1}{9c}$ in all pairs of $\{U'_{i,j} | 1 \leq j \leq h\}$, in which case we denote this color by k_i and remove from $\hat{G}$ all edges in $[V_i]^2$.

Note that for each $i < i'$ such that $\hat{G}$ has both edges and non-edges between V_i and $V_{i'}$, the density of the pair $U_i, U_{i'}$ (in M) is between $\frac{1}{3}$ and $\frac{2}{3}$, and so the regularity of this pair ensures that the density of $U'_i, U'_{i'}$ is between $\frac{2}{9}$ and $\frac{7}{9}$. Thus there exist two colors $k_{i,i'}$, $k'_{i,i'}$ such that between $U'_{i,j}$ and $U'_{i',j'}$ the density of the edges between them which are colored with $k_{i,i'}$, as well as the density of the non-edges between them which are colored with $k'_{i,i'}$, is at least $\frac{1}{9c}$ (due to the regularity of $U_i, U_{i'}$ as a pair of vertex sets of the $2c$-chart defined above, we can choose $k_{i,i'}$ and $k'_{i,i'}$ which satisfy that the density of the edges colored with $k_{i,i'}$ as well as the density of the non-edges colored with $k'_{i,i'}$ is at least $\frac{2}{9c}$ between U'_i and $U'_{i'}$). Also, if the subgraph of $\hat{G}$ between V_i and $V_{i'}$ is a complete bipartite graph then $k_{i,i'}$ as above exists (as the density of $U_i, U_{i'}$ is at least $\frac{1}{3}$ and so the density of $U'_i, U'_{i'}$ is at least $\frac{2}{9}$), and if it is an edgeless bipartite graph then $k'_{i,i'}$ as above exists.

The coloring $\mathcal{D}$ of $\hat{G}$ is constructed as follows. For a pair v, v' of vertices, we find i, i' such that $v \in V_i$ and $v' \in V_{i'}$. If $i = i'$, we color the pair by k_i. Otherwise, assume without loss of generality that $i < i'$. In this case, the pair is colored by $k_{i,i'}$ if it is an edge of $\hat{G}$, and by $k'_{i,i'}$ if it is not an

edge of $\hat{G}$ (since M is a model for $\tilde{G}$, and thus also for $\hat{G}$, only the values of $k_{i,i'}$ and $k'_{i,i'}$ which were defined above are used).

If a member K of $\mathcal{F}$ with h vertices appears in $\hat{G}$ with identical colors according to $\mathcal{D}$, let $\{v_1, \ldots, v_h\}$ be the vertices of $\hat{G}$ that span it, and let $\{i_1, \ldots, i_h\}$ be such that $v_j \in V_{i_j}$ for $1 \leq j \leq h$. Lemma 5.2 now guarantees the existence of $u_j \in U'_{i_j,j}$ such that the subgraph of M induced by $\{u_1, \ldots, u_h\}$ is identical to K and also has identical colors according to $\mathcal{C}$, which contradicts the assumption that $\mathcal{C}$ is an $\mathcal{F}$-pair-coloring of M. Thus, $\mathcal{D}$ is an $\mathcal{F}$-pair-coloring of $\hat{G}$. $\square$

Theorem 6.2 *For every $\epsilon > 0$ and every family of pair-colored graphs $\mathcal{F}$ there exist $\delta = \delta(\epsilon, \mathcal{F})$ and $t = t(\epsilon, \mathcal{F})$, such that for every graph G with n vertices which is ϵ-far from being $\mathcal{F}$-pair-colorable, at least $\delta\binom{n}{t}$ of the sets of t vertices of G span induced subgraphs which are not $\mathcal{F}$-pair-colorable.*

Proof: We define $\mathcal{E}$ by $\mathcal{E}(0) = \frac{1}{2}\epsilon$ and $\mathcal{E}(r) = \gamma_{6.1}(r, \mathcal{F})$ for $r > 0$; we define a function $\mathcal{S}$ by $\mathcal{S}(r) = s_{6.1}(r, \mathcal{F})$. We set

$$\delta = \delta_{3.2}(3\epsilon^{-1}, 1, \mathcal{S}, \mathcal{E})$$

and

$$t = t_{3.2}(3\epsilon^{-1}, 1, \mathcal{S}, \mathcal{E}).$$

Given a graph G with n vertices which is ϵ-far from being $\mathcal{F}$-pair-colorable, we use Lemma 3.2 to find $\tilde{G}$ which differs from G in less than $\frac{1}{2}\epsilon\binom{n}{2}$ places, and an equipartition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ where $3\epsilon^{-1} \leq k \leq S_{3.2}(3\epsilon^{-1}, 1, \mathcal{S}, \mathcal{E})$, such that at least $\delta\binom{n}{t}$ of the induced subgraphs of G are models for $\tilde{G}$ as in the formulation of this lemma.

Since G is ϵ-far from being $\mathcal{F}$-pair-colorable, and because $k \geq 3\epsilon^{-1}$, there exist no graph $\hat{G}$ which differs from $\tilde{G}$ only on $\bigcup_{i=1}^{k}[V_i]^2$ and is $\mathcal{F}$-pair-colorable (as such a graph would differ from G in less than $\epsilon\binom{n}{2}$ places), and so Lemma 6.1 ensures that all the above induced subgraphs are not $\mathcal{F}$-pair-colorable. $\square$

Corollary 6.3 *For any fixed family $\mathcal{F}$ of c-pair-colored subgraphs, the property of a graph being $\mathcal{F}$-pair-colorable is testable.*

Proof: Let $\delta(\epsilon, \mathcal{F})$ and $t(\epsilon, \mathcal{F})$ be as in Theorem 6.2. Consider now the algorithm which queries about all pairs of a uniformly random set of t' vertices of the input graph G, with t' chosen so that $(1-\delta)^{t'/t} \leq \frac{1}{3}$, and accepts if and only if the subgraph of G induced by this random set is $\mathcal{F}$-pair-colorable. It is not hard to see that an $\mathcal{F}$-pair-colorable graph is always accepted, while a graph which is ϵ-far from being $\mathcal{F}$-pair-colorable is rejected with probability at least $\frac{2}{3}$. $\square$

7 Concluding remarks

Testability for other combinatorial objects

Using appropriate versions of the Regularity Lemma, testability of certain properties can be proven for other combinatorial structures related to graphs. For example, testability of first order properties of type "$\exists\forall$" can be proven in the context of digraphs and in the context of g-charts for a fixed $g > 1$ as well, using arguments similar to those found in [2] and variants of the Regularity Lemma suitable for directed graphs, starting with that of [7]. The version of this paper that appears as the ECCC technical report TR00-083 (2000) also includes a sketch of how to extend both the positive and the negative results of [2] to tournaments (these results do not appear here because the amount of space that their full proof would have required does not merit their inclusion). Testability of other, more generalized, coloring properties for these structures can be proven as well using methods analogous to those of Section 4 and Section 6.

The definition of the model structures (as in model graphs) has to be modified in each case; as an example, consider the following definition for tournaments. In this context $d(A, B)$ denotes the number of edges directed from a vertex of A to a vertex of B divided by $|A||B|$, and a pair A, B of vertex sets is called regular with respect to the tournament if it is regular with respect to the graph whose vertex set is $A \cup B$ and whose edge set consists of all the tournament edges that point from A to B.

Definition 7 (Model tournaments) *Given a tournament $\tilde{T}$ and a partition $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ of its vertices, an (h, s, t)-**based** (η, ϵ)-**model for** T **over** $\mathcal{A}$ is a tournament M with a partition $\mathcal{M} = \{W_{i,j} | 1 \leq i \leq k, 1 \leq j \leq h\}$ of its vertex set, satisfying the following.*

- *$s \leq |W_{i,j}| \leq t$ for all $1 \leq i \leq k$ and $1 \leq j \leq h$.*
- *If $h > 1$, then for every fixed $1 \leq i \leq k$, all pairs $W_{i,j}, W_{i,j'}$ for $1 \leq j < j' \leq h$ are ϵ-regular. Moreover, if $h > 1$, then for every $1 \leq j < j' \leq h$ the density of the edges pointing from $W_{i,j}$ to $W_{i,j'}$ is at least $\frac{1}{2}$, and the subtournament of T spanned by V_i is transitive.*
- *For every $1 \leq i < i' \leq k$ and $1 \leq j, j' \leq h$, the pair $W_{i,j}, W_{i',j'}$ is an ϵ-regular pair. Moreover, for every fixed $1 \leq i < i' \leq k$, one of the following three cases occurs. Either all the pairs $W_{i,j}, W_{i',j'}$ (where j and j' vary over $\{1, \ldots, h\}$) satisfy $\eta \leq d(W_{i,j}, W_{i',j'}) \leq 1 - \eta$; or all edges of $\tilde{T}$ between V_i and $V_{i'}$ point from V_i to $V_{i'}$, and $\eta \leq d(W_{i,j}, W_{i',j'})$ for all j, j'; or all edges of $\tilde{T}$ between V_i and $V_{i'}$ point from $V_{i'}$ to V_i, and $\eta \leq d(W_{i',j'}, W_{i,j})$ for all j, j'.*

It would be interesting to investigate the testability of first order properties of type "$\exists\forall$" of vertex ordered graphs; properties which use the order relation over vertices but not the "sequel" function seem more accessible in that respect.

The situation is also far from clear in the context of first order hypergraphs properties, since an appropriate hypergraph version of the Regularity Lemma is not known at the present. This presents yet another motivation for finding such a version.

A little more about vertex coloring with restrictions

Section 4 proves testability of a rather restrictive notion of vertex coloring with restrictions. With a little more work, testability of being $(\alpha, \mathcal{F})$-colorable for any $\alpha = (\alpha_1, \ldots, \alpha_c)$ satisfying a given set of weak linear inequalities can be proven. For certain strong inequalities, such as "$\alpha_1 > 0$", a reduction to another colorability problem (with a larger set of colors) can be performed, in a similar manner to the reduction in [2] of first order properties of type "$\exists\forall$" to colorability properties. It seems that a rather wide notion of colorability with restrictions can be proven to be testable.

Coloring pairs with restrictions and beyond

Even simple properties, such as the property of a graph G with n vertices having at least $\alpha\binom{n}{2}$ edges (for a fixed α), cannot be described in terms of coloring notions without explicitly introducing restrictions on the number of appearances of each color. It is not surprising that coloring notions by themselves do not capture the essence of all testable graph properties, because many testable properties do not have one-sided tests, as proven in [12], while colorability notions seem to have such tests, as is the case with $\mathcal{F}$-pair-colorability. Note however that the tests constructed for (vertex) $(\alpha, \mathcal{F})$-colorability are not, and cannot be, one-sided.

If we define $(\alpha, \mathcal{F})$-pair-colorings and define appropriately a notion of pair-colorability with restrictions, all graph properties proven in [12] to be testable become instances of this definition. It seems that some approximation of the testability of $(\alpha, \mathcal{F})$-pair-colorability may be provable. However, such a proof using (some extension of) the current methods would be very involved.

Since the proof in [12] gives much better bounds on the dependence of the number of queries an ϵ-test makes as a function of ϵ, a proof of the testability of pair-colorability with restrictions is significant only if it brings one closer to a full characterization of all testable graph properties. However, it is not clear whether this notion even captures the simple property of a graph G with n vertices containing at least $\alpha\binom{n}{3}$ distinct triangles (for a fixed α), which seems to be testable.

In fact, even an "upper bound" proof that all the "natural" testable properties are expressible by some notion of $\mathcal{F}$-(l-tuple)-colorability with restrictions would be welcome, despite the fact that testability of all such properties is currently far from certain.

A little more about non-testable properties

The non-testable first order tournament property here and the non-testable graph property described in [2] are based on the idea that existence of a global isomorphism is hard to test locally. Specificly, the property I' stating that a graph G with n vertices consists of two disjoint copies of the same graph is essentially proven in [2] to be non-testable even if $o(\sqrt{\log n})$ queries are allowed.

It is provable that also querying about all pairs of a randomly chosen set of $o(\sqrt{n})$ vertices is not enough, using the theorem from [13] about canonical testers together with Yao's method for proving lower bounds (rather than the counting argument used in [2]). It would be interesting to find out the size of the random set of vertices which allows a construction of a test of I' in this extended sense, and also how many queries it takes to test for I' by an adaptive algorithm, whose queries are not restricted to the pairs of a pre-chosen random vertex set.

Finally, it would be interesting to find natural non-testable graph properties other than those based on graph isomorphism. It would also be interesting to find out more about the expressive power of all first order properties in the context of property testing and indistinguishability, e.g. which "natural" (say, definable by Second Order Logic) graph properties are not indistinguishable from any first order property.

References

[1] N. Alon, R. A. Duke, H. Lefmann, V. Rödl and R. Yuster, The algorithmic aspects of the Regularity Lemma, *Journal of Algorithms* 16 (1994), 80–109.

[2] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy, Efficient testing of large graphs, *Combinatorica* 20 (2000), 451–476.

[3] N. Alon, M. Krivelevich, I. Newman and M. Szegedy, Regular languages are testable with a constant number of queries, *SIAM Journal on Computing* 30 (2001), 1842–1862.

[4] N. Alon and J. Spencer, *The Probabilistic Method*, Wiley (1992).

[5] B. Bollobás, *Extremal Graph Theory*, Academic Press, New York (1978).

[6] A. Czumaj and C. Sohler, Abstract combinatorial programs and efficient property testers, *Proceedings of the* 43rd *IEEE FOCS* (2002), to appear.

[7] A. Czygrinow, S. Poljak and V. Rödl, Constructive quasi-Ramsey numbers and tournament ranking, *SIAM Journal on Discrete Mathematics* 12 (1999), 48–63.

[8] E. Fischer, The art of uninformed decisions: A primer to property testing, *The Bulletin of the European Association for Theoretical Computer Science* 75 (2001), 97-126.

[9] E. Fischer and I. Newman, Testing of matrix properties, *Proceedings of the* 33rd *ACM STOC* (2001), 286–295.

[10] E. Fischer and I. Newman, Functions that have read-twice constant width branching programs are not necessarily testable, *Proceedings of the* 17th *CCC* (2002), 73–79.

[11] A. M. Frieze and R. Kannan, The Regularity Lemma and approximation schemes for dense problems, *Proceedings of the* 37th *Annual IEEE FOCS* (1996), 12–20.

[12] O. Goldreich, S. Goldwasser and D. Ron, Property testing and its connection to learning and approximation, *Journal of the ACM* 45 (1998), 653–750 (A preliminary version has appeared in the proceedings of the 37th FOCS, 1996).

[13] O. Goldreich and L. Trevisan, Three theorems regarding testing graph properties, *Proceedings of the* 42nd *IEEE FOCS (2001)*, 460–469.

[14] W. T. Gowers, Lower bounds of tower type for Szemerédi's Uniformity Lemma, *Geometric and Functional Analysis* 7 (1997), 322–337.

[15] J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, In: *Combinatorics, Paul Erdös is Eighty*, Vol II (D. Miklós, V. T. Sós, T. Szönyi eds.), János Bolyai Math. Soc., Budapest (1996), 295–352.

[16] I. Newman, Testing of functions that have small width branching programs, *Proceedings of the* 41st *IEEE FOCS* (2000), 251–258.

[17] V. Rödl and R. Duke, On graphs with small subgraphs of large chromatic number, *Graphs and Combinatorics* 1 (1985), 91–96.

[18] D. Ron, Property testing (a tutorial), In: *Handbook of Randomized Computing* (S. Rajasekaran, P. M. Pardalos, J. H. Reif and J. D. P. Rolim eds), Kluwer Press (2001).

[19] R. Rubinfeld and M. Sudan, Robust characterization of polynomials with applications to program testing, *SIAM Journal of Computing* 25 (1996), 252–271.

[20] E. Szemerédi, Regular partitions of graphs, In: *Proc. Colloque Inter. CNRS* No. 260 (J. C. Bermond, J. C. Fournier, M. Las Vergnas and D. Sotteau eds.), 1978, 399–401.

[21] A. C. Yao, Probabilistic computation, towards a unified measure of complexity. *Proceedings of the* 18th *IEEE FOCS* (1977), 222–227.