MC-FINITENESS OF RESTRICTED SET PARTITION FUNCTIONS

Y. FILMUS, E. FISCHER, J.A. MAKOWSKY, AND V. RAKITA

ABSTRACT. A sequence $s(n)$ of integers is MC-finite if for every $m \in \mathbb{N}$ the sequence $s^m(n) = s(n) \bmod m$ is ultimately periodic. We discuss various ways of proving and disproving MC-finiteness. Our examples are mostly taken from set partition functions, but our methods can be applied to many more integer sequences.

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1. Introduction

1.1. **Goal of this paper.** Given a sequence of integers $s(n)$ with some combinatorial interpretation, one wonders what can be said about the sequence $s(n)$. Ideally, we would like to have an explicit formula for $s(n)$, or some recurrence relation with coefficients being constant or polynomial in n. Second best is an asymptotic description of $s(n)$.

We could instead look at the sequence $s^m(n) \equiv s(n) \bmod m$ and try to describe $s^m(n)$. If for every modulus m the sequence $s^m(n)$ is ultimately periodic, we say that $s(m)$ is *MC-finite*. We consider MC-finiteness a legitimate topic in the study of integer sequences. MC-finiteness appears under this name only since the publication of [35] in 2010. Without its name, the concept appears in the literature before, but rarely, e.g., under the name of *supercongruence* [1, 3]. The four substantial monographs on integer sequences published after 2000 do not mention the concept at all, see [15, 38, 38, 40].

All the sequences we discuss in this paper appear in *The On-Line Encyclopedia of Integer Sequences, OEIS, https://oeis.org/*, [23], with a number starting with A. We give these numbers with the first mention of the sequence, and list them also at the end of the paper. Needless to say, our methods also apply to many other entries in OEIS.

This paper grew out of our attempts to show that the sequence $B_r(n)$ of restricted Bell numbers (only listed in OEIS for $r = 2, A005493$ and $r = 3, A005494$) and $S_r(n, k)$ of restricted Stirling numbers of the second kind $A143494 - A143496$ introduced in [10] are MC-finite.

The purpose of this paper is two-fold. Its first part is mostly expository and written with the intent to popularize the study of MC-finiteness for researchers in Integer Sequences. However, the statements that the examples chosen are MC-finite have not, to the best of our knowledge, been stated before in the literature. We have chosen our examples in order to familiarize the reader with the two general methods to establish MC-finiteness. The first is *logical methods*, pioneered by C. Blatter and E. Specker, [8, 49, 9], and further developed by two of the authors of this paper (EF and JAM), [18, 20]. The second is a *combinatorial method* to prove MC-finiteness, also first suggested by E. Specker in [49], and later independently by G. Sénizergues [47], but only made precise in [11]. This method is based on the existence of finitely many mutual poynomial recurrence relations over $\mathbb{Z}$ used to define the integer sequence. In a separate paper, these methods are applied to infinitely many integer sequences arising from finite topologies [34].

In this paper we investigate MC-finiteness and counterexamples thereof of integer sequences derived from counting various unrestricted and restricted set partitions and transitive relations. Among the unrestricted cases we look at the Bell numbers $B(n)$, $A000110$, and the Stirling numbers of the second kind $S(n, k_0)$, $A000453$. We also discuss the number of linear quasi-orders (pre-orders) $LQ(n)$, $A000670$, the number of quasi-orders (pre-orders) $Q(n)$, $A000798$, the number of partial orders $P(n)$, $A001035$, and the number of transitive relations $T(n)$, $A006905$, on the set $[n]$. The numbers $LQ(n)$ are called *ordered Bell numbers* or *Fubini numbers*, often denoted in the literature by $a(n)$ and also by $F(n)$. For the unrestricted cases the results are seemingly new, or at least have not been stated before, but are simple consequences of growth arguments and the logical method due to C. Blatter and E. Specker [9, 49], the *Specker-Blatter Theorem*.

Typical restricted cases, first introduced by A. Broder [10] and further studied in [5], are the Stirling numbers of the second kind $S_{A,r}(n, k)$, which count the partitions of $[n + r]$ into $k + r$ blocks such that the elements $i \leq r$ are all in different blocks and the size of each block is in $A \subseteq \mathbb{N}$. For $r = 2$ see $A143494$. The Bell numbers

$B_{A,r}(n)$ are defined as $\sum_k S_{A,r}(n,k)$, see $A005493$ for $r = 2$ and $A005494$ for $r = 3$. The same restrictions can also be imposed on Stirling numbers of the second kind $S_{A,r}(n,k)$, and on all the unrestricted cases above. For the restricted cases, the results are new and require non-trivial extensions of the Specker-Blatter Theorem. The Catalan numbers $A000108$ also have an interpretation as set partitions. They count the number of non-overlapping partitions, see [46, Theorem 9.4] and [31, Chapter 10]. Although this can be viewed as a restricted version of the Bell numbers, our results do not apply to this case, as we shall explain later.

1.2. **Outline of the paper.** In Section 2 we introduce C-finiteness and its modular variant MC-finiteness. In Section 3 we discuss the methods for proving and disproving C-finiteness and MC-finiteness, and in Section 4 we present immediate consequences of the logical method for set partitions without positional restrictions and without restrictions on size of the blocks. The three sections have tutorial character, although the MC-finiteness of the examples has not been stated before in the literature. In Sections 5 and 6 we discuss set partitions with positional restrictions and restrictions on size of the blocks, and how new logical tools are used to obtain C-finiteness and MC-finiteness in these cases. We conclude the main part of the paper with Section 7, where we present our conclusions and suggestions for further research, and in Section 8 we list the numbers of the discussed OEIS-sequences. There are four appendices. In Appendix A we discuss larger classes of polynomial recursive sequences and weaker versions of MC-finiteness. In Appendix B we prove a special case of the main theorem from [20] which suffices for our results in Section 6. In Appendix C we give the details for proving C-finiteness of restricted Stirling numbers of the second kind. Finally, in Appendix D, we give an explicit computation of $S_A(n,k)$.

2. C-finite and MC-finite sequences of integers

A sequence of integers $s(n)$ is *C-finite*[1] if there are constants $p, q \in \mathbb{N}$ and $c_i \in \mathbb{Z}, 0 \leq i \leq p-1$ such that for all $n \geq q$ the linear recurrence relation

$$s(n+p) = \sum_{i=0}^{p-1} c_i s(n+i), n \geq q,$$

holds for $s(n)$. C-finite sequences have limited growth, see e.g. [15, 29]:

Proposition 1. *Let s_n be a C-finite sequence of integers. Then there is $c \in \mathbb{N}^+$ such that for all $n \in \mathbb{N}$, $a_n \leq 2^{cn}$.*

Actually, a lot more can be said, see [22], but we do not need it for our purposes.

To prove that a sequence $s(n)$ of integers is not C-finite, we can use Proposition 1. To prove that a sequence $s(n)$ of integers is C-finite, there are several methods: One can try to find an explicit recurrence relation, one can exhibit a rational generating function, or one can use a method based on model theory as described in [19, 17]. The last method will be briefly discussed in Section 6.4 and further explained in Appendix C. It is referred to as method FM.

A sequence of integers $s(n)$ is modular C-finite, abbreviated as *MC-finite*, if for every $m \in \mathbb{N}$ there are constants $p_m, q_m \in \mathbb{N}^+$ such that for every $n \geq q_m$ there is a linear recurrence relation

$$s(n+p_m) \equiv \sum_{i=0}^{p_m-1} c_{i,m} s(n+i) \bmod m$$

[1] These are also called constant-recursive sequences or linear-recursive sequences in the literature.

with constant coefficients $c_{i,m} \in \mathbb{Z}$. Note that the coefficients $c_{i,m}$ and both p_m and q_m generally do depend on m.

We denote by $s^m(n)$ the sequence $s(n)$ mod m.

Proposition 2. *The sequence $s(n)o$ is MC-finite iff $s^m(n)$ is ultimately periodic for every m.*

Proof. MC-finiteness implies periodicity. The converse is from [45]. □

Clearly, if a sequence $s(n)$ is C-finite it is also MC-finite with $r_m = r$ and $c_{i,m} = c_i$ for all m. The converse is not true, there are uncountably many MC-finite sequences, but only countably many C-finite sequences with integer coefficients, see Proposition 4 below.

Examples 3.

(i) *The Fibonacci sequence is C-finite.*
(ii) *If $s(n)$ is C-finite it has at most simple exponential growth, by Proposition 1.*
(iii) *The Bell numbers $B(n)$ are* not C-finite*, but are* MC-finite.
(iv) *Let $f(n)$ be any integer sequence. The sequence $s_1(n) = 2 \cdot f(n)$ is ultimately periodic modulo 2, but not necessarily MC-finite.*
(v) *Let $g(n)$ be any integer sequence. The sequence $s_2(n) = n! \cdot g(n)$ is MC-finite.*
(vi) *The sequence $s_3(n) = \frac{1}{2}\binom{2n}{n}$ is not MC-finite: $s_3(n)$ is odd iff n is a power of 2, and otherwise it is even (Lucas, 1878). A proof may be found in* [27, Exercise 5.61] *or in* [49].
(vii) *The Catalan numbers $C(n) = \frac{1}{n+1}\binom{2n}{n}$ are not MC-finite, since $C(n)$ is odd iff n is a Mersenne number, i.e., $n = 2^m - 1$ for some m, see* [31, Chapter 13].
(viii) *Let p be a prime and $f(n)$ monotone increasing. The sequence*
$$s(n) = \begin{cases} p^{f(n)} & n \neq p^{f(n)} \\ p^{f(n)} + 1 & n = p^{f(n)} \end{cases}$$
is monotone increasing but not ultimately periodic modulo p, hence not MC-finite.

Proposition 4. *(i) There are uncountably many monotone increasing sequences which are MC-finite, and uncountably many which are not MC-finite.*
(ii) Almost all integer sequences are not MC-finite.

Proof. (i) follows from Examples 3 (v) and (viii). (ii) is shown in Proposition 37 in Appendix A. □

Although we are mostly interested in MC-finite sequences $s(n)$, it is natural to check in each example whether the sequence $s(n)$ is also C-finite. In most examples the answer is negative. However, Theorem 33 shows that for restricted Stirling numbers of the second kind are all C-finite. We show this via a general method, Theorem 32, without exhibiting a generating function like in the classical case for $S(n, k)$.

3. How to prove and disprove MC-finiteness

3.1. **Polynomial recurrence relations.** In his 1988 paper [49, Page 144], E. Specker notes the following:

> In many known cases, [MC-finiteness] is a consequence of polynomial recurrence relations
> $$f(n) = \sum_{i=1}^{d} P_i(n) f(n-i)$$
> where P_i are polynomials in $\mathbb{Z}[x]$.

For $f(n) = n!$ this is obvious.

Definition 1. *(i) An integer sequence $s(n)$ is* holonomic over $\mathbb{Z}$ *if there exist polynomials $P_i \in \mathbb{Z}[x]$ with $P_1, P_k \neq 0$ such that*

$$s(n) = \sum_{i=1}^{k} P_i(n)s(n-i)$$

(ii) An integer sequence $s(n)$ is polynomially recursive (PRS) over $\mathbb{Z}$ *if there exist $k \in \mathbb{N}$ integer sequences $s_i(n), 1 \leq i \leq k$ with $s(n) = s_1(n)$ and polynomials $P_i \in \mathbb{Z}[x_1, \ldots, x_k]$ such that the following mutual recursion holds:*

$$s_i(n+1) = P_i(s_1(n), \ldots, s_k(n)), i = 1, \ldots k$$

(iii) An integer sequence $s(n)$ is PRS over $\mathbb{Z}$ and n *if the polynomials also involve n as an additional variable. In other words $P_i \in \mathbb{Z}[x_1, \ldots, x_k, y]$ and*

$$s_i(n+1) = P_i(s_1(n), \ldots, s_k(n), n), i = 1, \ldots k$$

Actually, (ii) and (iii) are equivalent.

We note that, if $s(n)$ is an integer sequence which is polynomially recursive over $\mathbb{Z}$ and n then $s(n)$ is holonomic over $\mathbb{Z}$.

In fact, the following is true:

Theorem 5. *If $s(n)$ is an integer sequence which is polynomially recursive over $\mathbb{Z}$ and n then $s(n)$ is MC-finite. In particular, this is true also for integer sequences $s(n)$ holonomic over $\mathbb{Z}$.*

The proof is given in Appendix A. There we also briefly discuss weaker properties than MC-finiteness, where the modular recurrence holds only for almost all $m \in \mathbb{N}^+$.

Remarks 6. *(i) In general, holonomic sequences are defined over fields $\mathbb{F}$ rather than the ring $\mathbb{Z}$. A good reference is* [29, Chapter 7]*. A theorem related to Theorem 5 for holonomic sequences can be found in* [1, Theorem 7]*, see also* [3]*.*

(ii) In [11]*, polynomially recursive sequences are defined for rational numbers rather than integers, and the polynomials are in $\mathbb{Q}[x_1, \ldots, x_k]$.*

The following examples, besides (v), are from [11].

Examples 7. *(i) The sequences $a(n) = n!$ with $a(n) = n \cdot a(n-1)$ and $a(0) = 1$ is holonomic over $\mathbb{Z}$. It is obviously MC-finite.*

(ii) The sequence $a(n) = 2^{2^n}$ is polynomialy recursive with $a(0) = 2$ and $a(n) = a(n-1)^2$. It is not holonomic, since every holonimc sequence $a(n)$ is bounded by some $2^{p(n)}$ for some polynomial $p(n)$, see [24]*. It is easy to see that it is MC-finiteR, but it is also MC-finite by the Specker-Blatter Theorem below, as it counts the number of ways one can interpret a unary predicate on $[n]$.*

(iii) The Catalan numbers C_n are holonomic: $(n+2)C_{n+1} = (4n+2)C_n$. They are not holonomic over $\mathbb{Z}$, since they are not MC-finite. Furthermore, they are not polynomially recursive even if we allow rational numbers.

(iv) The sequence n^n is not polynomialy recursive, but it is MC-finite by the Specker-Blatter Theorem below.

(v) We show in Appendix A that the sequence A086714 given by $a(0) = 4, a(n+1) = \binom{a(n)}{2}$ is not MC-finite but periodic modulo every odd number.

MC-finite sequences are closed under various arithmetic operations.

Proposition 8. *Let $a(n), b(n)$ be MC-finite sequences and $c \in \mathbb{Z}$.*

(i) Then $c \cdot a(n), a(n) + b(n), a(n) \cdot b(n)$ are MC-finite.

(ii) *If additionally, $b(n) \in \mathbb{N}^+$ and tends to infinity, $a(n)^{b(n)}$ is also MC-finite.*
(iii) *Let $A \subseteq \mathbb{N}^+$ be non-periodic and $a(n) = 2$ be a constant, hence MC-finite, sequence. The sequence*

$$b(n) = \begin{cases} 1 & n \in A \\ n! + 1 & n \notin A \end{cases}$$

is MC-finite and oscillates. However $a(n)^{b(n)}$ is not MC-finite.

3.2. **A definability criterion.** In order to prove that a sequence $s(n)$ is MC-finite one can also use a method due to E. Specker and C. Blatter from 1981 [8, 9, 49]. It uses logical definability as a sufficient condition. We denote by FOL first order logic, by MSOL monadic second order logic, and by CMSOL the logic MSOL augmented with modular counting quantifiers. Details on the definition of CMSOL are given in Section 6.1. In its simplest form, the Specker Blatter Theorem can be stated as follows:

Theorem 9 (Specker-Blatter Theorem)**.** *Let $S_\phi(n)$ be the number of binary relations R on a set $[n]$ which satisfy a given formula $\phi \in$ CMSOL. $S_\phi(n)$ is MC-finite, or equivalently, $S^m_\phi(n)$ is ultimately periodic for every m.*

The original Specker-Blatter Theorem was stated for classes of structures with a finite set of binary relations definable in Monadic Second Order Logic MSOL. It also works with unary relations added. The extension to CMSOL is due to [18]. This method is abbreviated in the sequel by SB.

3.3. **Comparing the methods.** If one proves MC-finiteness for an integer sequence directly, the proof may be sometimes straightforward, but also sometimes tricky, and not applicable to other sequences. In contrast to this, Theorems 5 and 9 are meta-theorems. They only require to check for some structural data about the sequence $s(n)$, recurrence relations or logical definability. However, these meta-theorems are only existence theorems, without explicitly giving the required coefficients $c_{i,m}$ which show MC-finiteness.

Examples 10. *We note that the two meta-theorems cannot always be applied to the same integer sequences.*

(i) *The sequence $s(n) = n^n$ counts the number of unary functions (as binary relations) from $[n]$ to $[n]$, which is* FOL*-definable, but it is not polynomially recursive, as shown in* [11]*. However, MC-finiteness can also be established directy without much effort.*
(ii) *There are polynomially recursive sequences over $\mathbb{Z}$ (hence MC-finite) which grow as fast as 2^{2^n}, e.g., the sequence $a(0) = 2, a(n+1) = a(n)^2$ satisfies $a(n) = 2^{2^n}$. However, counting the number of k binary relations on $[n]$ is bounded by 2^{kn^2}. Hence, Theorem 9 cannot be applied. Again, MC-finiteness can also be established directy without much effort.*
(iii) *The class of regular simple graphs is not* CMSOL*-definable. For a general method for proving non-definability in* CMSOL*, see* [36]*. Hence Theorem 9 cannot be applied to the sequence* A295193*, which counts the number of regular simple graphs on n labelled nodes. In contrast to this, r-regular graphs are* FOL*-definable, hence Theorem 9 can be applied easily to the sequence $RG(n, r)$ which counts the number of labelled r-regular graphs. The existence of recurrences for fixed r is disussed in* [39] *and the references cited therein. For $r = 2, 3$ this is* A110040*. Recurrences for $r = 0, 1, 2$ are found easily. For $r = 3, 4$ explicit recurrences were published in* [43, 44]*, and for $r = 5$ in* [26]*. The recurrence for $r = 5$ is linear but very long. In* [25]*, it is shown that*

$RG(n,r)$ is holonomic (P-recursive) for every $k \in \mathbb{N}^+$. We have not checked whether $RG(n,r)$ is holonomic over $\mathbb{Z}$. In [44] it is shown that $RG(n,4)$ is polynomially recursive, but the equations given there do not show that $RG(n,4)$ is polynomially recursive over $\mathbb{Z}$. It seems that Theorem 9 is the most suitable method to show that for each r the sequence $RG(n,r)$ is MC-finite.

We will use an extension to CMSOL, MSOL extended by modular counting quantifiers, from [17], and a new extension which allows the use of hard-wired constants and is described in Section B.

Clearly, $S_\phi(n)$ is computable by brute force, given ϕ and n. In [49], it is mentioned that $S^m_\phi(n) = S_\phi(n) \bmod m$ can be computed more efficiently, but no details are given. Only the special case for $Q^m(n)$ is given, where $Q(n)$ is the number of quasi-orders on $[n]$.

4. Immediate consequences of the Specker-Blatter Theorem

4.1. The Bell numbers $B(n)$.

The Bell numbers $B(n)$ count the number of partitions of the set $[n]$. This is the same as counting the number of equivalence relations on $[n]$, which is expressible by an FOL-formula. Therefore, we get immediately from Theorem 9 that:

Theorem 11. *The Bell numbers $B(n)$ are MC-finite.*

The Bell numbers do satisfy some known congruences. For $m = p$ a prime, they satisfy the Touchard congruence

$$B(p+n) \equiv B(n) + B(n+1) \mod p.$$

However, this is not enough to establish MC-finiteness.

The Bell numbers are not C-finite, because they grow too fast. The following estimate is due to [14, 7].

Proposition 12. *For every $n \in \mathbb{N}^+$*

$$\left(\frac{n}{e \ln n}\right)^n \leq B(n).$$

Furthermore, for every $\epsilon > 0$ there is $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$

$$B(n) \leq \left(\frac{n}{e^{1-\epsilon} \ln n}\right)^n.$$

Better estimates are known, see [22, Proposition VIII.3], but are not needed here. Another way to see that Bell numbers are not C-finite is by noticing that they are not holonomic, [30]. There, and in [2], some variations of Bell numbers are also studied:

Definition 2. *(i) $B(n)_{k,m}$ counts the number of partitions of $[n]$ which have k blocks modulo m.*
(ii) $B(n)^{\pm} = B(n)_{0,2} - B(n)_{1,2}$ which are the Uppuluri-Carpenter numbers A000587.
(iii) $B(n)^{bc}$ counts the number of bicolored partitions *of $[n]$, i.e., the partitions of $[n]$ where the blocks are colored with two non-interchangeable colors C_1, C_2, A001861.*

Theorem 13. *The sequences $B(n), B(n)_{k,m}, B(n)^{\pm}, B(n)^{bc}$ are not holonomic, hence not C-finite, but they are MC-finite.*

Proof. That they are not holonomic is shown in [30], and in [2]. To see that they are MC-finite, we apply Theorem 9.

(i) $B(n)_{k,m}$ is definable in CMSOL. We say that there is a set $X \subseteq [n]$ which intersects every block in exactly one element, and $|X| = k \mod m$.

(ii) $B(n)^{\pm}$ is the difference of two MC-finite sequences, hence MC-finite.
(iii) $B(n)^{bc}$ counts the number of binary and unary relations E, C_1, C_2 on $[n]$ such that E is an equivalence relations, $C_1, C_2 \subseteq [n]$ partition $[n]$, and each of them is closed under E.

□

4.2. **Counting transitive relations.** The Bell numbers $B(n)$ count the number of equivalence relations $E(n)$ on a set $[n]$. Similarly we can look at the number of linear quasi-orders (linear pre-orders) $LQ(n)$, the number of quasi-orders (pre-orders) $Q(n)$, the number of partial orders $P(n)$, and the number of transitive relations $T(n)$ on the set $[n]$. These integer sequences were analyzed in [42]. They are all definable in FOL, and we have

Proposition 14. $B(n) = E(n) \leq LQ(n) \leq P(n) \leq Q(n) \leq T(n)$.

Proof. $E(n) \leq LQ(n)$: We can turn an equivalence relation into a linear quasi-order by linearly ordering the equivalence classes.

$LQ(n) \leq P(n)$: Each linear quasi-order can be made into a partial order by replacing every set of mutually equi-comparable elements in a linear quasi-order with an anti-chain.

$P(n) \leq Q(n)$: Each partial order is also a quasi-order.

$Q(n) \leq T(n)$: Each quasi-order is transitive. □

Hence we get using the Specker-Blatter Theorem and Proposition 14:

Theorem 15. *The sequences $B(n) = E(n), LQ(n), P(n), Q(n)$ and $T(n)$ are MC-finite but not C-finite.*

4.3. **Stirling numbers of the second kind.** Let $S(n, k)$ be the number of partitions of $[n]$ into k non-empty blocks. $S(n, k)$ is also known as the Stirling number of the second kind. Clearly,

$$B(n) = \sum_k S(n, k).$$

Theorem 16. *For fixed $k = k_0$ the sequence $S(n, k_0)$ is C-finite, and hence MC-finite.*

This can be seen by observing that $S(n, k_0)$ has a rational generating function, see [27, 7.47].

$$\sum_{n=0}^{\infty} S(n, k_0)x^n = \frac{x^{k_0}}{(1-x)(1-2x)\cdots(1-k_0x)}.$$

4.4. **Lah numbers $Lah(n)$, $A001286$.** If we modify the Stirling numbers of the second kind $S(n, k)$ such that the elements in the blocks of the partition are ordered between them, we arrive at the somewhat less known Lah number $Lah(n, k)$, $A001286$, introduced by I. Lah in [32, 33] in the context of actuarial science. Good references for Lah numbers are [27, 12]. The Lah numbers are also coefficients expressing rising factorials $x^{(n)}$ in terms of falling factorials $x_{(n)}$.

Proposition 17.

$$x^{(n)} = \sum_{k=1} Lah(n, k)x_{(k)} \text{ and } x_{(n)} = \sum_{k=1} Lah(n, k)x^{(k)}$$

In [28] six proofs of Proposition 17 are given. Furthermore, $Lah(n) = \sum_k Lah(n, k)$.

$Lah(n)$ counts the number of linear quasi-orders on $[n]$, hence $Lah(n) = LQ(n)$, and $Lah(n,k)$ counts the number of linear quasi-orders on $[n]$ with k sets of equi-comparable elements. Two elements u, v in a quasi-order are *equi-comparable* if both $u \leq v$ and $v \leq u$. This is again definable in first order logic FOL.

There are explicit formulas:

Proposition 18.

$$Lah(n,k) = \frac{n!}{k!} \cdot \binom{n-1}{k-1} = \sum_{j=0}^{n} s(n,j)S(j,k) \tag{1}$$

$$Lah(n) = \sum_k Lah(n,k) = n! \sum_k \frac{1}{k!} \cdot \binom{n-1}{k-1} \tag{2}$$

where $s(n,j)$ are the Stirling numbers of the first kind, see [13].

There is also a recurrence relation:

$$Lah(n+1,k) = Lah(n,k-1) + (n+k)Lah(n,k) \tag{3}$$

But again this is not enough to establish C-finiteness or MC-finiteness, since it is a recurrence involving both n and k.

Theorem 19. *Both $Lah(n)$ and $Lah(n,k_0)$ are MC-finite but not C-finite.*

Proof. It follows directly from Equation (1), and also from Equation (3), that for $k = k_0$ fixed the sequence $Lah(n,k_0)$ is not C-finite. MC-finiteness again follows using Theorem 9. □

Note however that the recurrence relation given in Equation (3) does not have constant coefficients.

4.5. **Summary so far.** Table 1 summarizes the results which are direct consequences of the growth arguments or non-holonomicity (NH) and the Specker-Blatter Theorem 9 (SB).

Series	C-finite	Proof	Theorem	MC-finite	Proof	Theorem
$S(n) = B(n)$	no	Growth	15	yes	SB	11
$S(n,k_o)$	yes	gen.fun	16	yes	gen.fun	16
$B(n)^{\pm}$	no	NH	13	yes	SB	13
$B(n)^{bc}$	no	NH	13	yes	SB	13
$LQ(n)$	no	Growth	15	yes	SB	15
$Q(n)$	no	Growth	15	yes	SB	15
$P(n)$	no	Growth	15	yes	SB	15
$T(n)$	no	Growth	15	yes	SB	15
$Lah(n) = LQ(n)$	no	Growth	19	yes	SB	19
$Lah(n,k_0)$	no	Growth	19	yes	SB	19

TABLE 1. Direct consequences of the Specker-Blatter Theorem

5. RESTRICTED SET PARTITIONS

The new results of this paper concern C-finiteness and MC-finiteness for restricted versions of set partitions. We have two kinds of restrictions in mind. The first are *positional restrictions* which impose conditions on the positions of the elements of $[n]$ where $[n]$ is equipped with its natural order. The second are *size restrictions* which impose conditions on the size of the blocks or the number of the blocks.

5.1. **Global positional restrictions.**

Definition 3. *Let A and B be two blocks of a partition of $[n]$.*

(i) *A and B are* crossing *if there are elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 < b_1 < a_2 < b_2$ or $b_1 < a_1 < b_2 < a_2$.*
(ii) *Let $\min A, \max A, \min B, \max B$ the smallest and the largest elements in A and B. A and B are* overlapping *if $\min A < \min B < \max A < \max B$ or $\min B < \min A < \max B < \max A$.*
(iii) *If A and B are overlapping they are also crossing, but not conversely.*
(iv) *The number $B(n)^{nc}$ of non-crossing set partitions on $[n]$ is one of the interpretations of the Catalan numbers,* [46].
(v) *The Bessel number $B(n)^B$ (A006789) is the number of non-overlapping set partitions on $[n]$,* [21].

The Catalan numbers $C(n)$ are not holonomic and not MC-finite. In [2] it is shown that the Bessel numbers $B(n)^B$ are not holonomic. Are the Bessel numbers $B(n)^B$ MC-finite? The positional restrictions here are *global* in the sense that they involve all of the elements of $[n]$ with their natural order. For non-holonomic integer sequences $s(n)$ that count the number of set partitions subject to global positional restrictions, we have currently no tools to decide whether they are MC-finite or not.

Next, we look at *local* positional restrictions one can impose on Stirling and Lah numbers, [10, 51, 41, 6, 5]. They are local because they only put restrictions on the positions of a fixed number of elements of $[n]$ with their natural order.

5.2. **Local positional and size restrictions.**

Recall that we denote by $[n]$ the set $\{1, 2. \ldots, n\}$. We denote by $S_r(n,k)$ the number of partitions of $[n+r]$ into $k+r$ non-empty blocks with the additional condition that the first r elements are in distinct blocks. The elements $1, \ldots, r$ are called *special elements* and the partitions where the first r elements are in distinct blocks are called r-partitions. When dealing with definability we view the special elements as *hard-wired* constants, i.e., constant symbols $a_i, 1 \leq i \leq r$ with a fixed interpretation by elements of $[n+r]$.

We define $S_r(n) = B_r(n)$ by

$$S_r(n) = \sum_k S_r(n,k).$$

$Lah_r(n,k)$, $A143497$, and $Lah_r(n)$ are defined analogously, with the condition that $a_1 < a_2 < \ldots < a_r$ are in different blocks. [41, 48].

Let $A \subseteq \mathbb{N}$. We denote by $S_{A,r}(n) = B_{A,r}(n)$, $S_{A,r}(n,k)$, $Lah_{A,r}(n)$ and $Lah_{A,r}(n,k)$ the number of corresponding partitions where every block has its size in A.

For $r = 0$, in the absence of special elements, we just write $S_A(n) = B_A(n)$, $S_A(n,k)$, $Lah_A(n)$ and $Lah_A(n,k)$.

A set $A \subseteq \mathbb{N}$ is *(ultimately) periodic* if there exist $p, n_0 \in \mathbb{N}^+$ such that for all $n \in \mathbb{N}$ ($n \geq n_0$) we have $n \in A$ iff $n + p \in A$. In other words, the characteristic function $\chi_A(n)$ of A is ultimately periodic in the usual sense, $\chi_A(n) = \chi_A(n+p)$ ($n \geq n_0$). Analogous definitions can be made for $LQ(n)$, denoted by $LQ_{A,r}$, and also called r-Fubini sequences, with OEIS-number $A232472$.

5.3. **Main results for restricted set partitions.**

Our results for restricted set partitions are summarized in Tables 2, 3, 4 and 5 below. FM refers to the proof method of [19, 17]. SB* refers to the extension of the Specker-Blatter Theorem to allow a fixed finite set of special elements as hard-wired constants.

Series	C-finite	Proof	Theorem	MC-finite	Proof	Theorem
$S_A(n) = B_A(n)$	no	Growth	24	yes	SB*	21
$S_A(n, k_0)$	yes	FM	33	yes	FM	33
$Lah_A(n) = LQ_A(n)$	no	Growth	26	yes	SB*	21
$Lah_A(n, k_0)$	no	Growth	26	yes	SB*	21

Table 2. With ultimately periodic A only

Series	C-finite	Proof	Theorem	MC-finite	Proof	Theorem
$S_r(n) = B_r(n)$	no	Growth	24	yes	SB*	28
$S_r(n, k_0)$	yes	FM	33	yes	FM	33
$Lah_r(n, k_0)$	no	Growth	26	yes	SB*	28

Table 3. With hard-wired constants only

Series	C-finite	Proof	Theorem	MC-finite	Proof	Theorem
$S_{A,r}(n) = B_{A,r}(n)$	no	Growth	22	yes	SB*	28
$S_{A,r}(n, k_0)$	yes	FM	33	yes	FM	33
$Lah_{A,r}(n, k_0)$	no	Growth	22	yes	SB*	28

Table 4. With ultimately periodic A and hard-wired constants

These results also hold for $LQ_{A,r}$, the r-Fubini numbers, and other similarly defined sequences.

Series	C-finite	Proof	Theorem	MC-finite	Proof	Theorem
$B(n)^B$	no	NH	[2]	???	???	—
$B(n)^{nc} = C(n)$	no	NH	[46]	no	[46, Theorem 9.4]	[2]

Table 5. With global positional restrictions

6. Proofs for the restricted cases

For the analysis of MC-finiteness in the restricted cases we need some additional tools.

6.1. **Ultimate periodicity of A.** Recall that a formula with a *modular counting quantifier* $C_{b,m}x\phi(x)$ is true in a structure $\mathfrak{B}$ if the cardinality of the set of elements in $\mathfrak{B}$ which satisfy $\phi(x)$, satisfies

$$|\{a \in B : \phi(a)\}| \equiv b \mod m.$$

CMSOL is the logic obtained from MSOL by extending it with all the modular counting quantifiers $C_{b,m}$. In [18] the Specker-Blatter Theorem was extended to hold for CMSOL, as already stated in Theorem 9. CMSOL is also needed to prove the following lemma:

Lemma 20. *Let A be ultimately periodic and $\psi(x)$ be a formula of* CMSOL*. Then there is a sentence $\psi_A \in$* CMSOL *such that in every finite structure $\mathfrak{B}$ we have*

$$\mathfrak{B} \models \psi_A \text{ iff } |\{b \in B : \psi(b)\}| \in A$$

To be checked

Proof. If $A = A_{a,m} = \{n \in \mathbb{N} : n \equiv a \mod m\}$ the formula ψ_A is the sentence $C_{a,m}x\psi(x)$.

Next we observe that if A is ultimately periodic there are finitely many $a_1, \ldots, a_k$ and q such that $A = \bigcup_{i=0}^{k} A_i$ with $A_0 \subseteq [q]$ and $A_i = \{n > q : n \equiv a_i \mod m\}$. We proceed in steps:

(i) $\exists^{\geq k} x\psi(x) := \exists x_1, \ldots, x_k \bigwedge_{i=1}^{k} \psi(x_i) \wedge \bigwedge_{1 \leq i < j \leq k} (x_i \neq x_j)$ says that there are at least k elements that satisfy $\psi(x)$.

(ii) $\exists^{=k} x\psi(x) := \exists^{\geq k} x\psi(x) \wedge \neg\exists^{\geq k+1} x\psi(x)$ says that there are exactly k such elements.

(iii) $\psi A_0 := (\exists^{<q} x\psi(x) \to \bigvee_{j \in A_0} \exists^{=j} x\psi(x))$ says that if the number of elements satisfying $\psi(x)$ is less than q then the number of such elements has exactly one of the cardinalities in A_0.

(iv) $\psi A_i := \exists^{\geq q+1} x\psi(x) \wedge C_{a,m} x\psi(x)$ says that if the number of elements satisfying $\psi(x)$ is bigger or equal than q then the number such elements equals a_i mod m.

(v) $\psi(x)_A := \bigvee_{i=0}^{k} \psi A_i(x)$ is the required formula.

□

Theorem 9 together with Lemma 20 gives immediately:

Theorem 21. *Assume that A is ultimately periodic. Then the sequences $B_A(n) = S_A(n), Lah_A(n)$ and $Lah_A(n, k_0)$ are MC-finite.*

6.2. **Growth arguments.** We first discuss growth arguments for $B_A(n) = S_A(n), Lah_A(n)$ and $Lah_A(n, k_0)$.

Theorem 22. *Let $A \subseteq \mathbb{N}$ be infinite and ultimately periodic. Then $B_A(n) = S_A(n), Lah_A(n)$ and $Lah_A(n, k_0)$ are not C-finite.*

Proof. First we prove it for $B_A(n)$ and $A = A_m = \{n \in \mathbb{N} : n \equiv 0 \mod m\}$. Let $P_1, \ldots, P_k$ be a partition of $[n]$. We replace in each P_i every element by m elements. This gives us a partition of $[mn]$ with each block of size in A_m. Hence

$$P_A(mn) \geq P(n) \geq \left(\frac{n}{e \ln n}\right)^n$$

or, equivalently,

$$P_A(n) \geq P(n/m) \geq \left(\lfloor \frac{n/m}{\rfloor} e \ln \lfloor n/m \rfloor\right)^{\lfloor n/m \rfloor}$$

which still grows superexponentially.

Next we assume that $A = A_{k,a,m} = \{n \in \mathbb{N} : n \equiv a \mod m, n \geq k\}$. We proceed as before, but additionally add $mr + a$ elements to each block, for r large enough. Finally, we note that for every infinite (ultimately) periodic set A there is a set $A_{k,a,m}$ for some $k, a, m \in \mathbb{N}^+$ such that $A_{k,a,m} \subseteq A$

For $Lah_A(n)$ and $Lah_A(n, k_0)$ we proceed similarly using Proposition 18. □

Next we discuss growth for $Lah(n, k_0)$, $Lah(n) = \sum_k Lah(n, k)$ and $Lah_r(n, k_0)$. We have seen in Proposition 12 that

$$\left(\frac{n}{e \ln n}\right)^n \leq B(n) \leq \left(\frac{n}{e^{1-o(1)} \ln n}\right)^n$$

We now show

Lemma 23. $B_r(n) \geq B(n)$

Proof. Every partition of $[n]$ gives rise to at least one partition of $[n + r]$ where the first r elements are in distinct blocks containing only one element. □

From Proposition 1, 14 and 23 we get:

Theorem 24. *The sequences $B(n)$ and $B_r(n)$ are not C-finite.*

Lemma 25. *For k_0, r fixed, the Lah number $Lah(n, k_0)$ satisfy the following:*
(i) $Lah(n, k_0) = \binom{n-1}{k_0-1}\frac{n!}{k_0!}$,
(ii) $Lah(n) \geq Lah(n, k_0)$, *and*
(iii) $Lah_r(n, k_0) \geq Lah(n, k_0)$.

Proof. (i) is from [32, 33]. (ii) follows from (i), and (iii) is proved like Lemma 23. □

This gives immediately

Theorem 26. *Let k_0 be fixed. The sequences $Lah(n, k_0)$, $Lah(n) = \sum_k Lah(n, k)$ and $Lah_r(n, k_0)$ are not C-finite.*

6.3. **Hard-wired constants.** Recall that a constant is *hard-wired* on $[n]$ if its interpretation is fixed.

The Specker-Blatter Theorem is originally proved for classes of structures with a finite number of binary relations. It is false for one quaternary relations [16]. It was announced recently that it is also false for one ternary relation, [20].

The Specker-Blatter Theorem remains true when adding a finite number of unary relations. This is so because a unary relation $U(x)$ can be expressed as a binary relation $R(x, x)$ which is false for $R(x, y)$ when $x \neq y$.

Adding constants comes in two flavours, with variable interpretations, or hard-wired. Assume we want to count the number of unary predicates P on $[n]$ which contain the interpretation of a constant symbol c. There are n possible interpretations for c and 2^{n-1} interpretations for sets not containing c, hence $n2^{n-1}$ many such sets. However, if c is hard-wired to be interpreted as $1 \in [n]$, there are only 2^{n-1} many such sets.

Constants can be represented as unary predicates the interpretation of which is a singleton. If we do this, the Specker-Blatter Theorem holds, but we cannot model the r-Bell numbers like this. To prove that the r-Bell numbers are MC-finite one has to deal with r many hard-wired constants. Adding a finite number of hard-wired constants needs some work. In Appendix B we show how to eliminate a finite number of hard-wired constants for the case of $S_r(n)$. The proof generalizes. In [20] the more general version is proved:

Theorem 27. *Let τ_r be a vocabulary with finitely many binary and unary relation symbols, and r hard-wired constants. Let ϕ be a formula of* $\mathrm{CMSOL}(\tau_r)$. *Then $S_\phi(n)$ is MC-finite.*

Corollary 28. *The sequences $S_r(n) = B_r(n)$, $Lah_r(n, k_0)$, $S_{A,r}(n) = B_{A,r}(n)$, $Lah_{A,r}(n, k_0)$ are MC-finite.*

6.4. **Proving C-finiteness.** In this subsection we explain a special case of the method used in [19] to prove C-finiteness. It is based on counting partitions of graphs satisfying additional properties and computing these partitions for iteratively constructed graphs.

6.4.1. *Counting partitions with a fixed number of blocks.* Let $G = (V(G), E(G))$ be a graph, and $k_0 \in \mathbb{N}$. We look at partitions $P_1(G), \ldots, P_{k_0}(G)$ of $V(G)$ which can be described in first order logic FOL. The following are three typical examples:

Examples 29. *(i) The underlying sets of $G[P_i(G)]$ form a partition of $V(G)$ without further restrictions.*
(ii) For each $i \leq k_0$ the induced graph $G[P_i(G)]$ is edgeless (proper coloring).

(iii) Let $\mathcal{P}$ be a graph property. For each $i \leq k_0$ the set $G[P_i(G)]$ is in $\mathcal{P}$ ($\mathcal{P}$-coloring).

We look at the counting function

$$f_\phi(G) = |\{P_1(G), \ldots, P_{k_0}(G) : \phi(P_1(G), \ldots, P_{k_0}(G))\}|$$

defined using an FOL-formula ϕ.

Let $A \subseteq \mathbb{N}$ be an ultimately periodic set. We also look at the restricted counting function

$$f_{\phi,A}(G) = |\{P_1(G), \ldots, P_{k_0}(G) : \phi(P_1(G), \ldots, P_{k_0}(G)) \text{ and } |P_i(G)| \in A\}|.$$

We also allow graphs with a fixed number of distinct vertices, which may appear in the formula ϕ.

6.4.2. *Iteratively constructed graphs.*

Definition 4. *A k-colored graph is a graph G together with k sets $V_1, V_2, ..., V_k \subseteq V(G)$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$. A basic operation on k-colored graphs is one of the following:*

- *Add_i: add a new vertex of color i to G.*
- *$Recolor_{i,j}$: recolor all vertices with color i to color j in G.*
- *$Uncolor_i$: remove the color of all vertices with color i. Uncolored vertices cannot be recolored again.*
- *$AddEdges_{i,j}$: add an edge between every vertex with color i and every vertex with color j in G.*
- *$DeleteEdges_{i,j}$: delete all edges between vertices with color i and vertices with color j from G.*

A unary operation F on graphs is elementary if F is a finite composition of basic operations on k-colored graphs (with k fixed). We say that a sequence of graphs $\{G_n\}$ is iteratively constructed if it can be defined by fixing a graph G_0 and defining $G_{n+1} = F(G_n)$ for an elementary operation F.

Example 30. *The following sequences are iteratively constructed:*

- *The complete graphs K_n can be constructed using two colors: Fix G_0 to be the empty graph, and the operation F, given a graph G_n, adds a vertex with color 2, adds edges between all vertices with color 2 and color 1, and recolors all vertices with color 2 to color 1.*
- *The paths P_n can be constructed using 3 colors: Fix G_0 to be the empty graph, and the operation F, given a graph G_n, adds a vertex with color 3, adds edges between all vertices with colors 2 and 3, recolors all vertices with color 2 to color 1, and recolors all vertices with color 3 to color 2.*
- *The cycles $C_n, n \geq 3$ can be constructed by first constructing a path P_n where the first and the last element have colors 1 and 2 different from the remaining vertices. Then we connect the first and last element of P_n by an edge. This needs 5 colors, but is not iterative. To make it an iterative construction we proceed as follows. Given a cycle C_n with with two neighboring vertices of color 1 and 2, uncolor all the other vertices and remove the edge $(1,2)$. Then add a new vertex with color 3, make edges $(1,3)$ and $(3,2)$, uncolor the old vertices colored by 1, and then recolor 3 to have color 1.*

Remark 31. *In [19] there was an additional operation allowed*

- *Duplicate: Add a disjoint copy of G to G,*

assuming erroneously that Duplicate behaves like a unary operation on graphs. Although it looks like a unary operation on graphs, the sequence of graphs

$$G_0 = E_1, G_{n+1} = Duplicate(G_n)$$

grows too fast and does not fit the framework that the authors have envisaged in [19]*.*

6.4.3. *The FM method.* In this framework [19] proved the following:

Theorem 32 (The Fischer-Makowsky Theorem)**.** *Let G_n be an iteratively constructed sequence of graphs, $A \subseteq \mathbb{N}$ be ultimately periodic, and*

$$f_\phi(G_n) = |\{P_1(G_n), \ldots, P_{k_0}(G_n) : \phi(P_1(G_n), \ldots, P_{k_0}(G_n))\}|$$

and

$$f_{\phi,A}(G_n) = |\{P_1(G_n), \ldots, P_{k_0}(G_n) : \phi(P_1(G_n), \ldots, P_{k_0}(G_n)) \text{ and } |P_i(G_n)| \in A\}|,$$

where $\phi \in$ CMSOL. Then the sequences $f_\phi(G_n)$ and $f_{\phi,A}(G_n)$ are C-finite.

We now use Theorem 32 to prove:

Theorem 33. *Let A be ultimately periodic, $r, k_0 \in \mathbb{N}$. Then $S(n, k_0)$, $S_A(n, k_0)$, $S_r(n, k_0)$ and $S_{A,r}(n, k_0)$ are C-finite.*

Proof. It suffices to prove it for $S_{A,r}(n, k_0)$. The other cases can be obtained by setting $r = 0$ and/or $A = \mathbb{N}$.

We have to show that $S_{A,r}(n, k_0)$ is of the form $f_{\phi,A}(G_n)$.

We define an iteratively constructed sequence of graphs $G = (V(G), E(G), v_1, \ldots, v_r)$ with r distinct vertices as follows. $G_0 = (K_r, v_1, \ldots, v_r)$. $G_{n+1} = G_n \sqcup K_1$.

Now take $\phi(P_1, \ldots, P_{k_0}, v_1, \ldots, v_r)$ which says that the P_i's form a partition and for each $i \leq r$ the distinguished vertex v_i belongs to $P_i(G)$. □

Further details are given in Appendix C.

7. Conclusions and further research

In the first part of the paper we introduced MC-finiteness as a worthwhile topic in the study of integer sequences. We surveyed two methods of establishing MC-finiteness of such sequences. In Theorem 5, MC-finiteness follows from the existence of polynomial recurrence relations with coefficients in $\mathbb{Z}$. In Theorem 9, MC-finiteness follows from a logical definability assumption in Monadic Second Order Logic augmented with modular counting quantifiers CMSOL. We have compared the advantages and disadvantages of the methods, and we have used the logic method of Theorem 9 to give quick and transparent proofs of MC-finiteness.

In the second part of the paper we got similar results for locally restricted set partition functions like $B_{A,r}$. For this purpose the Specker-Blatter Theorem has to be extended in order to count labeled structures where a fixed number of special elements are in a certain configuration. In the case of $B_{A,r}$, A is a set of natural numbers and r is a natural number. $B_{A,r}$ counts the number of set partitions of $[n]$ where the first r elements are in different blocks and A indicates the possible cardinalities of the blocks of the partition. Such an extension is given in Theorem 27. A proof of a special case of this theorem is given in the appendix. The general case can be found in [20]. Our new results are summarized in Tables 2–5.

We did not investigate in depth whether MC-finiteness of the examples in Tables 2–5 can be established directly or by exhibiting suitable polynmial recurrence schemes, in order to apply Theorem 5.

Problem 1. *: Are the Bessel numbers $B(n)^B$ MC-finite?*

Problem 2. *: Find systems of mutual polynomial recurrences for all the examples in Tables 2–4.*

Instead of set partition functions we can also count the number of, say, partial orders where

(i) r special elements are in a particular CMSOL definable configuration, such as prescribed comparability and incomparability, and
(ii) A indicates the possible cardinalities of certain definable sets, such as antichains or maximal linearly ordered sets.

Our techniques allow us to show that counting such partial orders on $[n]$ results in MC-finite sequences.

In [49] it is suggested that counting the number of quasi-orders $Q^m(n)$ on $[i]$ modulo m is easier than finding the exact value of $Q(n)$.

Clearly, $S_\phi(n)$ is computable by brute force, given ϕ and n. In fact, for $\phi \in \mathrm{FOL}$ the problem is in $\sharp\mathbf{P}$. For $\phi \in \mathrm{CMSOL}$ it is in $\sharp\mathbf{PH}$, the analogue of $\sharp\mathbf{P}$ for problems definable in Second Order Logic, or equivalently, in the polynomial hierarchy. As noted in [37, Proposition 11], there are arbitrarily complex problems in $\mathbf{PH}$ already definable in MSOL. However, $S^m_\phi(n)$ is in $MOD_m\mathbf{P}$, respectively in $MOD_m\mathbf{PH}$, the corresponding modular counting classes introduced in [4]. It is still open how exactly $MOD_m\mathbf{P}$ is related to $\sharp\mathbf{P}$.

In [49], it is mentioned that $S^m_\phi(n) = S_\phi(n) \mod m$ can be computed more efficiently, but no details are given. Only the special case of $Q^m(n)$ is given, where $Q(n)$ is the number of quasi-orders on $[n]$.

Problem 3. *Given $\phi \in \mathrm{FOL}$ and m, find algorithms for computing $S_\phi(n)$ and $S^m_\phi(n)$ and determine upper and lower bounds for them. One may assume that n is encoded in unary.*

Problem 4. *Same as Problem 3 for $\phi \in \mathrm{CMSOL}$.*

Problem 5. *Inspired by the remarks above, the following might be a worthwhile project: Investigate the complexity classes $\sharp\mathbf{PH}$ and $MOD_m\mathbf{PH}$ and their mutual relationships.*

8. List of OEIS-sequences

A000108: Catalan numbers $C(n)$.
A000110: Bell numbers $B(n)$.
A000453: Stirling numbers of the send kind $S(n,k)$.
A000587: Uppuluri-Carpenter numbers $A000587$.
A000670: Number of linear quasi-orders (pre-orders) $LQ(n)$.
A000798: Number of quasi-orders (pre-orders) $Q(n)$.
A001035: Number of partial orders $P(n)$.
A001286: Lah numbers $Lah(n)$.
A001861: Bicolored partitions.
A005493: r-Bell numbers $B_{A,2}(n)$ for $r = 2$.
A005494: r-Bell numbers $B_{A,3}(n)$ for $r = 3$.
A006905: Number of transitive relations $T(n)$.
A086714: $a(0) = 4, a(n+1) = \binom{a(n)}{2}$.
A110040: Regular labeled graphs of degree 2 and 3.
A143494: r-Stirling numbers $S_{A,r}(n,k)$
A143497: r-Lah numbers $Lah_{A,r}(n)$.
A232472: r-Fubini numbers $LQ_{A,r}$ for $r = 2$.
A295193: Regular labeled graphs.

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Appendix A. More on MC-finiteness

A.1. **Polynomial recursive sequences.** A *polynomial recursive sequence* [11] is a mutual recurrence in which the recurrence relation is a polynomial. That is, we define d sequences in parallel by initial values $a_1(0), \ldots, a_d(0)$ and the recurrence

$$a_i(n+1) = P_i(a_1(n), \ldots, a_d(n)),$$

where P_i is a polynomial with rational coefficients. We will only consider recurrences for which $a_i(n) \in \mathbb{N}$ for all $i \in [d]$ and $n \geq 0$.

Theorem 34 ([11])**.** *Let m be a natural number which is relatively prime to all denominators of coefficients of the defining polynomials $P_1, \ldots, P_d$. Then the sequences $a_i(n) \bmod m$ are eventually periodic.*

Proof. Notice that

$$a_i(n+1) \bmod m = (P_i \bmod m)(a_1(n) \bmod m, \cdots a_d(n) \bmod m).$$

Thus the function $P \colon \mathbb{Z}_m^d \to \mathbb{Z}_m^d$ given by

$$P(x_1, \ldots, x_d) = ((P_1 \bmod m)(x_1, \ldots, x_d), \ldots, (P_d \bmod m)(x_1, \ldots, x_d))$$

satisfies

$$(a_1(n+1) \bmod m, \dots, a_d(n+1) \bmod m) = P(a_1(n) \bmod m, \dots, a_d(n) \bmod m).$$

Since $\mathbb{Z}_m^d$ is finite, if we start at $(a_1(0) \bmod m, \dots, a_d(0) \bmod m)$ and repeatedly apply P, we will eventually enter a cycle. $\square$

This result raises the following question: what happens for other m? It turns out that the theorem fails in general for such m.

Theorem 35. *Consider the following sequence A086714:*

$$a(n+1) = \binom{a(n)}{2}, \quad a(0) = 4.$$

The sequence $a(n) \bmod 2$ is not eventually periodic.

The same result holds (with the same proof) for any $a(0) \geq 4$, as well as for any recurrence of the form $a(n+1) = (a(n)+b)(a(n)+c)/2$, as long as b, c have different parities and $a(0)$ is chosen so that $a(n) \to \infty$.

A.2. **Proof of Theorem 5.** Let $\beta(n) = a(n) \bmod 2$. It is not hard to check that the sequence $\beta(n) \dots \beta(n+k-1)$ depends only on $a(n) \bmod 2^k$. It turns out that the opposite holds as well: we can determine $a(n) \bmod 2^k$ from $\beta(n) \dots \beta(n+k-1)$.

Lemma 36. *Let a_r, β_r be defined as above, except with the initial condition $a_r(0) = r$.*

For all $k \geq 1$, the function

$$\Phi_k(r) = \beta_r(0) \dots \beta_r(k-1)$$

is a bijection between $\{0, \dots, 2^k - 1\}$ and $(0,1)^k$.

For example, if $k = 3$, we get the following bijection:

$$\begin{array}{llll} \Phi_3(0) = 000 & \Phi_3(1) = 100 & \Phi_3(2) = 010 & \Phi_3(3) = 111 \\ \Phi_3(4) = 001 & \Phi_3(5) = 101 & \Phi_3(6) = 011 & \Phi_3(7) = 110 \end{array}$$

Proof. The proof is by induction on k. The result is clear when $k = 1$, so suppose $k > 1$.

The first bit of $\Phi_k(r)$ is the parity of r, and the remaining bits are $\Phi_{k-1}(s)$, where $s = \binom{r}{2} \bmod 2^{k-1}$. To complete the proof, we show that the mapping $r \mapsto s$ is 2-to-1, with the two preimages of every s having different parity.

Indeed, suppose that $\binom{a}{2} \equiv \binom{b}{2} \pmod{2^{k-1}}$ for $a, b \in \{0, \dots, 2^k - 1\}$. Then $a(a-1) \equiv b(b-1) \pmod{2^k}$, and so $2^k \mid a(a-1) - b(b-1) = (a-b)(a+b-1)$.

If a, b have the same parities then $a + b - 1$ is odd and so $2^k \mid a - b$. Since $a, b \in \{0, \dots, 2^k - 1\}$, in this case $a = b$.

If a, b have different parities then $a - b$ is odd and so $2^k \mid a + b - 1$, and so $b = 1 - a \bmod 2^k$ is uniquely defined, and has a parity different from a. $\square$

We can now prove Theorem 35. First, notice that $\binom{a}{2} > a$ for $a \geq 4$, and so $a(n) \to \infty$. Now suppose that the sequence β is ultimately periodic, say with period $\beta(N), \dots, \beta(N + \ell - 1)$. Lemma 36 implies that for every $k \geq 1$, the sequence $a(n) \bmod 2^k$ has period $a(N) \bmod 2^k, \dots, a(N + \ell - 1) \bmod 2^k$, and in particular, $a(N) \equiv a(N + \ell) \bmod 2^k$. Choosing k such that $2^k > a(N + \ell)$, we reach a contradiction.

A.3. **Normal sequences.** Let $s(n)$ be an integer sequence, and $b \in \mathbb{N}^+$. The sequence $s^b(n) = s(n) \bmod b$ is normal, that is, if we chunk it into substrings of length $\ell \geq 1$ then each of the b^ℓ possible strings of $[b]^\ell$ appear in $s^b(n)$ with equal limiting frequency. It is *absolutely normal* if it is normal for every b. The sequence $s^b(n) = s(n) \bmod b$ can be viewed as a real number r_b written in base b. A classical theorem from 1922 by E. Borel says that almost all reals are absolutely normal, [15]. The theorem below shows that MC-finite integer sequences are very rare.

Let PR_b be the set of integer sequences $s^b(n)$ with $s^b(n) = s(n) \bmod b$ for some integer sequence $s(n)$. PR_b is the projection of all integer sequences to sequences over $\mathbb{Z}_b$. We think of PR_b as a set of reals with the usual topology and its Lebesgue measure. Let $UP_b \subseteq PR_b$ be the set of sequences $s^b(n) \in PR_b$ which are ultimately periodic.

Proposition 37. *(i) Almost all reals are absolutely normal.*
(ii) $s(n)$ is MC-finite iff for every $b \in \mathbb{N}^+$ the sequence $s^b(n)$ is ultimately periodic
(iii) If $s^b(n)$ is normal for some b, then $s(n)$ is not MC-finite.
(iv) $UP_b \subseteq PR_b$ has measure 0.

Proving that a specific sequence is normal is usually difficult.

Here is a challenge:

Conjecture 1. *The binary sequence $\beta(n) = a(n)$* mod 2 *from Theorem 35 is normal with $b = 2$.*

Appendix B. Eliminating hard-wired constants

Let $\mathfrak{S}_r(n) = ([r+n], a_1, \ldots, a_r, E)$ be the structures on $[r+n]$ where E is an equivalence relation and the r elements $a_1, \ldots, a_r$ are in different equivalence classes. $S_r(n)$ counts the number of such structures on $[r+n]$.

Let $\mathfrak{E}_r(n)$ be a structure on $[n]$ which consists of the following:

(i) $E(x,y)$ is an equivalence relation on $[n]$;
(ii) There are r unary relations $U_1, \ldots, U_r$ on $[n]$;
(iii) The sets $U_i(x)$ are disjoint;
(iv) Each $U_i(x)$ is either empty or consists of exactly one equivalence class of E;

Let $E_r(n)$ be the number of such structures on $[n]$.

Lemma 38. *For every $r, n \in \mathbb{N}^+$ there is a bijection f between the structures $\mathfrak{E}_r(n)$ on $[n]$ and the structures $\mathfrak{S}_r(n)$ on $[r+n]$, hence we have $E_r(n) = S_r(n)$.*

Proof. Given a structure $\mathfrak{S}_r(n)$ we define $f(\mathfrak{S}_r(n))$ as follows:

(i) The universe of $f(\mathfrak{S}_r(n))$ is $\{r+1, \ldots, r+n\}$.
(ii) If for $i \leq r$ the set $\{i\}$ is a singleton equivalence class, we put $U_i = \emptyset$. If there is an equivalence class E_i which strictly contains i we put $U_i = E'_i = E_i - \{i\}$.
(iii) E' is the equivalence relation induced by E on $\{r+1, \ldots, r+n\}$.

Conversely, given a structure $E_r(n) = ([n], E, U_1, \ldots, U_r)$ we define $g(E_r(n))$ as follows:

(i) The universe of $g(E_r(n))$ is $[n+r]$ and the equivalence relation E' is defined by defining its equivalence classes.
(ii) If U_i is empty for some $i \geq n+1$ the singleton $\{i\}$ is an equivalence class of E'. If U_i is not empty, then the equivalence class of E' which contains i is $U_i \cup \{i\}$.
(iii) If C is an equivalence class of E such that $U_i \neq C$ for all $i \geq n+1$, then C is an equivalence class for E'.

It is now easy to check that f, g are bijections and g is the inverse of f. $\square$

Remarks 39. *(i) Clearly the class of structures $E_r(n)$ as defined here is* FOL-*definable. Hence we can apply the Specker-Blatter Theorem and conclude that $S_r(n)$ is MC-finite.*

(ii) If A is ultimately periodic then $S_{A,r}(n)$ is also MC-finite. To see this we note that for $S_{A,r}(n)$ all the equivalence classes C satisfy $|C| \in A$. This means that in a structure $\mathfrak{E}_{A,r}(n)$ the equivalence classes C satisfy $|C| \in A$, if they do not contain a U_i, and $|C| \in A'$ where $A' = \{a - 1 : a \in A\}$, otherwise. If A is ultimately periodic, so is A' and both are definable in CMSOL.

(iii) For the Lah numbers $L_r(n)$ and $L_{A,r}(n)$ we proceed likewise by replacing the equivalence relation by a linear quasi-order. For every i we add two further unary relations and the appropriate conditions in order to take care of the ordering of the special elements. Hence both $L_r(n)$ and $L_{A,r}(n)$ are MC-finite.

Appendix C. Proof of Theorem 32 and its applications

In order to prove Theorem 32 we use Theorem 40 below. For this we have to introduced thedefinition of CMSOL-definable graph polynomials.

C.1. CMSOL-definable graph polynomials.

Definition 5. *Let $\mathbb{Z}$ be the ring of integers. We consider polynomials over $\mathbb{Z}[\overline{x}]$. For an* CMSOL-*formula for graphs $\phi(\overline{v})$ with $\overline{v} = (v_1, \ldots, v_s)$, define $card_G(\phi)$ to be the cardinality of subsets of $V(G)^s$ defined by ϕ. The extended* CMSOL *graph polynomials are defined recursively. We first define the* extended CMSOL-monomials. *Let $\phi(\overline{v}) \in$* CMSOL. *An extended* CMSOL-*monomial is a term of one of the following possible forms:*

- $x^{card_G(\phi)}$ *where x is one of the variables of $\overline{x}$.*
- $x_{(card_G(\phi))}$ *i.e. the falling factorial of x.*
- $\binom{x}{card_G(\phi)}$.
- $\prod_{\overline{v} \in V(G)^s : \phi(v)} t(\overline{x})$ *where $t(\overline{x})$ is a term in $\mathbb{Z}[\overline{x}]$.*

The extended CMSOL graph polynomials *are obtained from the monomials by closing under finite addition and multiplication. Furthermore they are closed under summation over subsets of $V(G)$ of the form*

$$\sum_{U : \phi(U)} t$$

where ϕ is an CMSOL-*formula with free set variables U, and under multiplication over elements of $V(G)^s$ of the form*

$$\prod_{\overline{v} \in V(G)^2 : \phi(v)} t(\overline{x})$$

.

Theorem 40 (Theorem 1 [19])**.** *Let F be an elementary operation on graphs, $\{G_n : n \in \mathbb{N}\}$ an F-iterated sequence of graphs, and P an extended* CMSOL-*definable graph polynomial. Then $\{G_n\}$ is C-finite, i.e. there exist polynomials $p_1, p_2, ..., p_k \in \mathbb{Z}[\overline{x}]$ such that for sufficiently large n,*

$$P(G_{n+k+1}) = \sum_{i=1}^{k} p_i P(G_{n+i})$$

This proves Theorem 32.

C.2. **Proofs of C-Finiteness.** Now we give the deatiled proofs of Theorem 33.

Proposition 41. *Fix $k_0 \in \mathbb{N}$. Then $S(n, k_0)$ is a C-finite sequence.*

Proof. Let P be the graph property of cliques with at least one vertex, i.e. $P = \{K_n : n \geq 1\}$, and define $G_n = K_n$. Note that a P-coloring of G_n with k_0 colors is a partition of $V(G_n)$ into exactly k_0 non empty color classes, so $H_P(G_n, k_0) = S(n, k_0)$. We want to apply the Fischer-Makowsky theorem. First, note that the sequence G_n is iteratively constructible, see Example 30 or [19, Proposition 2]. Hence H_P is an extended CMSOL graph polynomial and we can use Theorem 40. $\square$

Proposition 42. *Fix $k_0 \in \mathbb{N}$. Then $S_r(n, k_0)$ is a C-finite sequence.*

Proof. Let P be the graph property of edgeless graphs with at least one vertex, i.e. $P = \{\overline{K_n} : n \geq 1\}$, and define $G_n = K_r \cup \overline{K_n}$. Note that a P-coloring of G_n with $k_0 + r$ colors is a partition of $V(G_n)$ into exactly $k_0 + r$ non empty color classes, such that every vertex in $V(K_r) \subseteq V(G_n)$ is in a different color class, so $H_P(G_n, k_0 + r) = S_r(n, k_0)$. We want to apply the Fischer-Makowsky theorem. First, note that the sequence G_n is iteratively constructible: put $G_0 = K_r$. Now given G_n, we construct G_{n+1} by adding a disjoint vertex. Hence H_P is again an extended CMSOL graph polynomial and we can use Theorem 40. $\square$

Proposition 43. *Let $A \subseteq \mathbb{N}$, and $k_0 \in \mathbb{N}$. Then $S_A(n, k_0)$ is a C-finite sequence if and only if A is ultimately periodic.*

Proof. First, note that $S_A(n, 1) = 1$ iff $n \in A$. Therefore, if A is not ultimately periodic, $S_A(n, 1)$ is not C-finite. On the other hand, assume A is ultimately periodic. Let P be the graph property of cliques with vertex size in A, i.e. $P = \{K_n : n \in A\}$, and define $G_n = K_n$. Note that a P-coloring of G_n with k_0 colors is a partition of $V(G_n)$ into exactly k_0 non empty color classes, with each color class with size in A. so $H_P(G_n, k_0) = S_A(n, k_0)$. We want to apply the Fischer-Makowsky theorem. As before, the sequence G_n is iteratively constructible. Hence H_P is again an extended CMSOL graph polynomial and we can use Theorem 40.

$\square$

Proposition 44. *Let $A \subseteq \mathbb{N}$, and $k_0 \in \mathbb{N}$. Then $S_{A,r}(n, k_0)$ is a C-finite sequence if and only if A is ultimately periodic.*

Proof. $S_A(n, 1) = 1$ iff $n \in A$. If A is not ultimately periodic, then also $S_A(n, 1)$ is not C-finite. Assume A is ultimately periodic. Let P be the graph property of edgeless graphs with vertex size in A, i.e. $P = \{\overline{K_n} : n \in A\}$, and define $G_n = K_r \cup \overline{K_n}$. Note that a P-coloring of G_n with $k_0 + r$ colors is a partition of $V(G_n)$ into exactly $k_0 + r$ non empty color classes with sizes in A, such that every vertex in $V(K_r) \subseteq V(G_n)$ is in a different color class, so $H_P(G_n, k_0 + r) = S_r(n, k_0)$. We want to apply the Fischer-Makowsky theorem. As before, the sequence G_n is iteratively constructible. Hence H_P is again an extended CMSOL graph polynomial and we can use Theorem 40. $\square$

Appendix D. An explicit computation of $S_A(n, k)$

Let $A \subseteq \mathbb{N}$. $S_A(n, k)$ counts the number of partitions of $[n]$ into k sets with cardinalities in A.

We shall compute $S_A(n, k)$ explicitly. For $A = \mathbb{N}^+$ this will give also an alternative way of computing $S(n, k)$, the Stirling numbers of the second kind. The method is reminiscent to [12, Theorem 8.6] or, in very different notation [50, Chapter 1, Exercise 45].

We introduce some suitable notation. Let $A \subseteq \mathbb{N}$. $S_A(n,k)$ counts the number of partitions of $[n]$ into k sets with cardinalities in A. Let $V(A,k)$ be the set of k-tuples $(L_1,\ldots,L_k)$ of elements of A ordered in non-decreasing order, with $\sum_{i=1}^{k} = n$, i.e.

$$V(A,k) = \{(l_1,l_2,...,l_k) \in A^k : 0 < l_1 \leq l_2 \leq ... \leq l_k, \sum_{i=1}^{k} l_i = n\}.$$

For $(l_1,l_2,...,l_k) \in V(A,k)$ define $g(m;l_1,l_2,...,l_k)$ to be the number of times m appears in the k-tuple $(l_1,l_2,...,l_k)$, and

$$f(l_1,l_2,...,l_k) = \prod_{m\in(l_1,l_2,...,l_k)} g(m;l_1,l_2,...,l_k)!.$$

Next we define inductively: $c_1 = n$, $c_{i+1} = c_i - l_i$, hence $c_i = n - \sum_{1}^{i-1} l_i$.

Theorem 45. *Let $A \subseteq \mathbb{N}$. Then*

$$S_A(n,k) = \sum_{(l_1,l_2,...,l_k)\in V(A,k)} \frac{1}{f(l_1,l_2,...,l_k)} \prod_{i=1}^{k} \binom{c_i}{l_i}$$

Proof. To partition $[n]$ into k sets with cardinalities in A, we proceed as follows: First, we select the cardinalities of the k sets. This corresponds to picking an element $(l_1,l_2,...,l_k) \in V(A,k)$. To construct a partition of n, we choose l_1 elements from $[n]$, then l_2 elements from $[n-l_1]$ etc. Finally, we divide by $f(l_1,l_2,...,l_k)$ to account for double counting of tuples with equal entries. □

Faculty of Computer Science Technion-Israel Institute of Technology, Haifa Israel
Email address: yuvalfi@technion.ac.il

Faculty of Computer Science Technion-Israel Institute of Technology, Haifa Israel
Email address: eldar@cs.technion.ac.il

Faculty of Computer Science Technion-Israel Institute of Technology, Haifa Israel
Email address: janos@cs.technion.ac.il

Faculty of Mathematics Technion-Israel Institute of Technology, Haifa Israel
Email address: vsevolod@campus.technion.ac.il