



הטכניון – מכון טכנולוגי לישראל  
Technion – Israel Institute of Technology

**ספריות הטכניון**  
*The Technion Libraries*

**בית הספר ללימודי מוסמכים ע"ש ארווין וג'ואן ג'ייקובס**  
*Irwin and Joan Jacobs Graduate School*

©

**All rights reserved**

*This work, in whole or in part, may not be copied (in any media), printed, translated, stored in a retrieval system, transmitted via the internet or other electronic means, except for "fair use" of brief quotations for academic instruction, criticism, or research purposes only.  
Commercial use of this material is completely prohibited.*

©

**כל הזכויות שמורות**

*אין להעתיק (במדיה כלשהי), להדפיס, לתרגם, לאחסן במאגר מידע, להפיץ באינטרנט, חיבור זה או כל חלק ממנו, למעט "שימוש הוגן" בקטעים קצרים מן החיבור למטרות לימוד, הוראה, ביקורת או מחקר. שימוש מסחרי בחומר הכלול בחיבור זה אסור בהחלט.*

# Using Property Testing for Efficient Detection of Nearly-Sorted Relations

Sagi Ben Moshe



# Using Property Testing for Efficient Detection of Nearly-Sorted Relations

Research Thesis

Submitted in Partial Fulfillment of the Requirements  
for the Degree of Master of Science

Sagi Ben Moshe

Submitted to the Senate of the  
Technion - Israel Institute of Technology

Adar 5770

Haifa

March 2010



The research thesis was done under the supervision of  
Assoc. Prof. Eldar Fischer  
in the Department of Computer Science.

## Acknowledgements

I am grateful to my supervisor - Eldar Fischer, for guiding me devotedly for the last two years, for his continued encouragement and for many invaluable suggestions in this work. I would also like to include my gratitude to my collaborators: Yaron Kanza, Arie Matsliah, Carl Staelin, Mani Fischer.

The generous financial help of the Technion and of the ERC-2007-StG is gratefully acknowledged



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
<b>3</b>	<b>Outline of the suggested query evaluation strategy</b>	<b>9</b>
<b>4</b>	<b>Exploiting Near Sortedness</b>	<b>11</b>
<b>5</b>	<b>Evaluating Other Operators</b>	<b>16</b>
<b>6</b>	<b>Testing for Near Sortedness</b>	<b>18</b>
<b>7</b>	<b>Discussion</b>	<b>23</b>
7.1	Testing for several values of $k$ and $\ell$ at once . . . . .	23
7.2	Distances larger than memory . . . . .	25
7.3	Recovering from a fail mode of the improved sort algorithm . . . . .	25
7.4	Solid State Drives . . . . .	26
7.5	Parallel scalability . . . . .	27
7.6	Preliminary experiments about natural occurrence of near-sortedness . . . . .	28
	<b>Bibliography</b>	<b>29</b>
<b>A</b>	<b>Appendix</b>	<b>32</b>
A.1	A simple sortedness test . . . . .	32
A.2	A sortedness test with better probe complexity . . . . .	34



## Abstract

Many relational operations are best performed when the relations are stored sorted over the relevant attributes (e.g. the common attributes in a natural join operation). However, generally relations are not stored sorted because it is expensive to maintain them this way (and impossible whenever there is more than one relevant sort key). Still, many times relations turn out to be *nearly-sorted*, where most tuples are close to their place in the order. This state can result from “leftover sortedness”, where originally sorted relations were updated, or were combined into interim results when evaluating a complex query. It can also result from weak correlations between attribute values. Currently, nearly-sorted relations are treated the same as unsorted relations, and when relational operations are evaluated for them, a generic algorithm is used. Yet, many operations can be computed more efficiently by an algorithm that exploits this near-ordering.

However, to consistently benefit from using such algorithms the system should also refrain from using the wrong algorithm for relations which happen not to be sorted at all. Thus, an efficient *test* is required, i.e., a very fast approximation algorithm for establishing whether a given relation is sufficiently nearly-sorted.

In this paper, we provide the theoretical foundations for improving query evaluation over possibly nearly-sorted relations. First we formally define what it means for a relation to be nearly-sorted, and show how operations over such relations, such as natural join, set operations and sorting, can be executed significantly more efficiently using an algorithm that we provide. If a relation is nearly-sorted enough, then it can be sorted using two sequential reads of the relation, and writing no intermediate data to disk. We then construct efficient probabilistic tests for approximating the degree of the near-sortedness of a relation without having to read an entire file. The role of our algorithms in a database management system setting is illustrated as soon as the theoretical foundation is laid out. Finally, we show how our approach can also benefit distributed systems and systems that use a solid-state drive.

This thesis is organized as follows. In Chapter 2 we provide the framework and define near-sortedness. In Chapter 3 we outline the suggested strategy. In Chapter 4 we describe in detail the algorithm for the efficient evaluation of the sorting operation on nearly-sorted relations, where in Chapter 5 we describe how other operations can be performed efficiently without sorting in advance. In Chapter 6 we present a test for determining whether relations are nearly-sorted. This is the version that is required for the strategy outlined in Chapter 3, a tolerant two-parameter test that estimates what can be performed on the relations with respect to the existing memory constraints.

Simpler restricted versions of this test appear in the appendix; they may be useful in some restricted contexts, and their analysis can serve as a warm-up to the analysis of Chapter 6. Finally, in Chapter 7 we discuss additional issues, as well as some possible extensions of our algorithms.

# Chapter 1

## Introduction

Typically, database query processors handle relations that are stored unsorted on the disk. When a processor needs to access the tuples of a relation in some specific order, this can be done through an appropriate index, if such an index exists, or by initially sorting the relation and then retrieving the tuples. Otherwise the system would have been required to constantly maintain the relations sorted, and this would cause updates and insertions to be inefficient. Additionally, if the setting is such that more than one order is relevant (*e.g.* if there is more than one index) then of course it is impossible to maintain the relation sorted for all of them.

However, when querying the data, there are operations whose evaluation is far more efficient if the relations are sorted. We will refer to such operations as *order-preferring operations*. For example, the natural join of two relations that are sorted on their join attributes can be done in a single pass over the relations, *i.e.*, a single sequential read of the files (assuming that every set of tuples with common values in the join attributes can fit into computer memory), whereas other join methods, such as nested-loop join, require more than one pass over at least one of the relations [5]. Additional order-preferring operations include the set operations (UNION, INTERSECT and EXCEPT) and operations whose implementation has to use sorting, *e.g.*, grouping, duplicate elimination (DISTINCT) and calculating aggregate functions. An index could help to evaluate some order-preferring operations, but it does not always exist for the attributes that we need, and creating and maintaining an index is costly.

A simple approach to executing order-preferring operations is to first sort the relations and then apply the operation. However, the sorting operation itself is expensive and may require several passes over the relations. In particular, we will show this approach is not optimal for the *nearly-sorted* relations that we will consider here.

We define nearly-sorted relations with respect to two parameters  $k$  and  $\ell$ . The parameter  $\ell$  tells us how rough is the sorting: a larger  $\ell$  means that tuples can be further away

from their in-place position, before they are considered to be out of place. The parameter  $k$  tells us how many tuples are completely out of place, that is, how many tuples need to be disregarded before the remaining relation conforms with the  $\ell$  parameter.

Nearly-sorted relations occur in various scenarios. Some examples are (1) a relation that has been stored as sorted and has been slightly updated (a small number of arbitrary updates are captured by  $k$ , and certain systemic updates are captured by  $\ell$ ); (2) a relation that has been created by a previous operation (such as a join) performed on sorted relations; (3) a relation that was sorted on one attribute (or on a set of attributes) can be nearly-sorted on another attribute due to naturally occurring correlations (*e.g.*, weight to height, apartment size to rental fee and so on).

If the given relation is nearly-sorted then the query processor can execute order-prefering operations very efficiently. Specifically, in Chapter 4 we show an algorithm that sorts nearly-sorted relations in at most two sequential reading passes. The algorithm requires memory of size roughly  $t(2k + \ell)$ , where  $t$  is the maximal size of a tuple and  $k, \ell$  are the near-sortedness parameters defined above (we also provide a simpler less efficient algorithm as a warm-up). In contrast, traditional methods (without prior conditions on the relations) require more than just two sequential passes over the relation, even just for sorting (multiway merge sort can usually be done in two “passes”, but one of them requires fully random access, which is much more expensive than a sequential pass). Further more, our improved algorithm (Algorithm 2) makes no writes at all to disk (apart from writing the output itself), a vast improvement over all previous algorithms (including multiway merge). The reason that we focus first on the sorting operation itself is not only because it is the easiest to explain among database operations, but also because it contains the core algorithm. In Chapter 5 we explain how to incorporate this algorithm into the execution of other operations, such as natural join (equijoin) and set operations (intersection, etc). The following example illustrates the savings.

**Example 1.** *Consider the natural join of two relations,  $R_1$  and  $R_2$ , that are nearly sorted on a single shared attribute, in the sense that at most  $k$  tuples of each relation violate the order of the tuples according to this attribute. A naive join of the relations will be to sort each relation and then to apply a sort-based join. For each relation, sorting it will require at least two phases of reading all the tuples of the relation and writing all the tuples to the disk (and possibly more phases if the reads have to be sequential). After sorting, another pass over the two relations will be needed for the join itself (in some cases the last pass of the sorting algorithms can be merged with this one).*

*However, knowing that the relations are nearly sorted allows to compute the join by merely two sequential readings of the relations and without writing temporary results to disk.*

Note that for relations that are much larger than computer memory, a nested-loop join (the sort-less option) will require more than two sequential readings of the relations. Similarly, ad-hoc creation of indexes for the join will be more expensive than two sequential reads of the relations and not viable in most cases.

In many cases, applying order-preferring operations on nearly-sorted relations can be done even when the number of tuples which violate the order is quite large.

**Example 2.** If in Example 1 both  $R_1$  and  $R_2$  have a size of 10 gigabyte and we use a memory of 2 gigabyte for the join, a computation of a join as described in Example 1 can be done even when approximately 10% of the tuples in each relation violate the order.

When in Example 1 the relations  $R_1$  and  $R_2$  are *not* nearly sorted, the described algorithm will fail due to lack of memory for keeping the order-violating tuples. In such a case, the system would need to use a generic join algorithm, *i.e.*, an algorithm that can join unsorted relations. Actually our algorithm can recover from a failing state and fall back to a less efficient one, but it is still best to efficiently test whether the given relations are nearly sorted and choose the appropriate algorithm in advance. Moreover, in order for the approach to be efficient, this test should only read a small part of the relation.

To test that the relations are nearly sorted we use ideas from the theory of property testing. In general, property testing refers to the following type of problems: Given the ability to perform local inspections (here, reading specific tuples) of a particular object (here, a relation), the goal is to determine whether the object has a predetermined property (here, one related to being sorted), or is far from having the property. The task should be performed by inspecting only a small part of the whole object, where a small probability of failure is allowed. See [3, 9, 10] for surveys on property testing.

In our case, we also require that the computational overhead will be small, and we deal with two parameters,  $k$  and  $\ell$ , rather than just measuring the distance from having the property (this distance would conform to our  $k$ ). Most importantly, we require our tests to be *tolerant*, because we actually want to guarantee acceptance for small enough non-zero values of  $k$  and  $\ell$ , and not only guarantee rejection for values that are too large, so as not to miss on any inputs for which our optimization is possible. Also, for practical applications the number of queries has not only to increase slowly with the input size  $n$ , but to depend not too badly on  $n/k$ , where the best possible (which we achieve here) is a linear dependence. Additionally, the test is non-adaptive, in that it is able to provide all the locations for the reads ahead of obtaining any answers. This allows modern operating systems to optimize the reading operations and compensate for possible seek times.

Since the entire relation should be considered by the test, even though not all the tuples of the relations are being read, the choice of which tuples to read must be prob-

abilistic. Traditionally, the probability of failure is taken to be at most  $\frac{1}{3}$ , but it can be made arbitrarily smaller by applying the test several times. In general, to guarantee a failure probability of at most  $\delta$ , we would need to increase the number of tuples being read by an  $O(\log(1/\delta))$  factor. For example, if a test requires  $\sqrt{n}$  inspections where  $n$  is the number of tuples in the relation, then for a relation of 1,000,000 tuples, 1,000 reads determine the property with probability of success  $\frac{2}{3}$ . By applying an amplification technique, 6,000 reads provide probability of success  $1 - (\frac{1}{3})^6$  which is approximately 0.999.

From the property-testing point of view, the testing algorithms that we develop here generalize the tolerant monotonicity tests that were developed in [1, 8]. The algorithms of [1, 8] can only distinguish between almost sorted arrays (in the sense that removing a few elements makes them completely sorted) and those that are far from being sorted. In contrast, our testers work with a more relaxed notion of sortedness, where an unbounded number of elements can be out of order, but not too far from their correct location. As we explain in Remark 2.4 below, there is no simple relation between these monotonicity notions that would allow us to use the testers of [1, 8] as they are. The analysis of the new tests is an important contribution of this thesis.

The thesis is organized as follows. In Chapter 2 we provide the framework and define near-sortedness. In Chapter 3 we outline the suggested strategy. In Chapter 4 we describe in detail the algorithm for the efficient evaluation of the sorting operation on nearly-sorted relations, where in Chapter 5 we describe how other operations can be performed efficiently without sorting in advance. In Chapter 6 we present a test for determining whether relations are nearly-sorted. This is the version that is required for the strategy outlined in Chapter 3, a tolerant two-parameter test that estimates what can be performed on the relations with respect to the existing memory constraints. Simpler restricted versions of the test appear in the appendix; they may be useful in some restricted contexts, and their analysis can serve as a warm-up to the analysis of Chapter 6. Finally, in Chapter 7 we discuss additional issues, as well as some possible extensions of our algorithms.

## Chapter 2

### Preliminaries

We consider a relation  $R$  as an ordered sequence of tuples  $t_1, \dots, t_n$ . For relations that are stored in the database, the order is determined according to the order by which tuples are accessed in a sequential read of the relation. We also refer to  $R$  as an array – we denote by  $R[i]$  the tuple  $t_i$ , and we say that  $t_i$  appears in *location*  $i$  of  $R$ . We denote by  $[n]$  the set  $\{1, \dots, n\}$  of the possible indices in  $R$ , and by  $[n, m]$  we denote the set  $\{n, \dots, m\}$  if  $m \geq n$ , or the empty set if  $m < n$ .

A *sort key* of  $R$  is a pair  $K = (A, \leq_K)$  of an attribute  $A$  and an anti-symmetric transitive relation  $\leq_K$ . A sort key defines a *desired order* for the tuples of  $R$ . In the desired order, for every two tuples  $t_i$  and  $t_j$ , when  $\pi_A(t_i) \leq_K \pi_A(t_j)$ , the tuple  $t_i$  should appear before the tuple  $t_j$ . We generally use  $R[i] \leq_K R[j]$  to denote  $\pi_A(t_i) \leq_K \pi_A(t_j)$ . When  $K$  is clear from the context, we simply say that the tuple  $t_i$  is lesser than or equal to the tuple  $t_j$  and denote this by  $R[i] \leq R[j]$ . We also denote by  $\text{MIN\_VAL}_K$  a tuple with the smallest possible value (with respect to  $\leq_K$ ) of an attribute  $A$ . Again, we may use simply  $\text{MIN\_VAL}$  whenever  $K$  is clear from the context.  $\text{MAX\_VAL}$  is defined similarly.

The definition of a sort key can be generalized in a natural way to the case where it consists of more than one attribute:  $K = ((A_1, \dots, A_k), (\leq_{K_1}, \dots, \leq_{K_k}))$ , where  $A_1, \dots, A_k$  are attributes in the schema of  $R$ . In such a case, the desired order of the tuples is defined using a lexicographic order.

A relation that complies with the desired order defined by the key is called *sorted*.

**Definition 2.1** (sorted relation). *A relation  $R$  of  $n$  tuples is sorted according to a sort key  $K$ , if for any two indices  $i, j$ , where  $1 \leq i < j \leq n$ , we have  $R[i] \leq_K R[j]$ .*

A relation  $R$  is  *$k$ -close to being sorted* when it is possible to remove from it  $k$  tuples to achieve a sorted relation. Or alternatively, when there exists a set of at most  $k$  tuples so that the relation is fully sorted outside of it.

**Definition 2.2** ( $k$ -close to being sorted). A relation  $R$  of  $n$  tuples is  $k$ -close to being sorted according to a sort key  $K$ , if there exists a set of indices  $I$ , where  $|I| \leq k$ , so that for any two indices  $1 \leq i < j \leq n$ , where  $i \notin I$  and  $j \notin I$ , we have  $R[i] \leq_K R[j]$ . If a relation is not  $k$ -close to being sorted, then we say that it is  $k$ -far from being sorted.

A relation  $R$  is  $\ell$ -globally sorted according to a sort key  $K$ , when for every two tuples in  $R$  that do not comply with the order defined by  $K$ , the difference between the locations of these two tuples is strictly smaller than  $\ell$ .

**Definition 2.3** ( $\ell$ -globally sorted). Given a positive integer  $\ell$ , a relation  $R$  of  $n$  tuples is  $\ell$ -globally sorted according to  $K$ , if for any two indices  $i, j \in [n]$ , where  $i \leq j - \ell$ , we have  $R[i] \leq_K R[j]$ .

**Remark 2.4.** Notice that a relation is 1-globally sorted if and only if it is 0-close to being sorted (which is equivalent to being sorted). But in general, there is no correspondence between these two notions. It is an easy exercise to construct: (1) a relation  $R$  which is both 2-globally sorted and  $n/2$ -far from being sorted; (2) a relation  $R$  which is not even  $(n - 2)$ -globally sorted, but is 1-close to being sorted.

The following final definition combines the relaxation of Definition 2.2 with the relaxation of Definition 2.3, and captures the notion of being nearly-sorted as it was discussed in the introduction.

**Definition 2.5** ( $(k, \ell)$ -nearly sorted). Given a nonnegative  $k$  and a positive  $\ell$ , we say that  $R$  is  $k$ -close to being  $\ell$ -globally sorted according to  $K$ , or  $(k, \ell)$ -nearly sorted, if there exists a set of indices  $I$  where  $|I| \leq k$ , so that for any two indices  $i, j \in [n]$  where  $i \leq j - \ell$ ,  $i \notin I$  and  $j \notin I$  we have  $R[i] \leq_K R[j]$ .

If a relation is not  $(k, \ell)$ -nearly sorted, then we say that it is  $k$ -far from being  $\ell$ -globally sorted. Note that for any  $k' \leq k$  and  $\ell' \leq \ell$ , a relation that is  $(k', \ell')$ -nearly sorted is also  $(k, \ell)$ -nearly sorted.

Our definitions deviate a little from the standard notion of  $k$ -closeness, which requires the tuples with indices in  $I$  to be replaced with alternative tuples for which sortedness holds for the entire relation. However, by [2, 4, 11], the two definitions are equivalent for many monotonicity-like properties, including all those defined here.

**Lemma 2.6.** Let  $E \subseteq [n]$  be a set of indices, whose removal makes  $R$   $\ell$ -globally sorted. There is a way of replacing the tuples in  $E$  with new ones (rather than dismissing them) that will make  $R$  into an  $\ell$ -globally sorted relation.



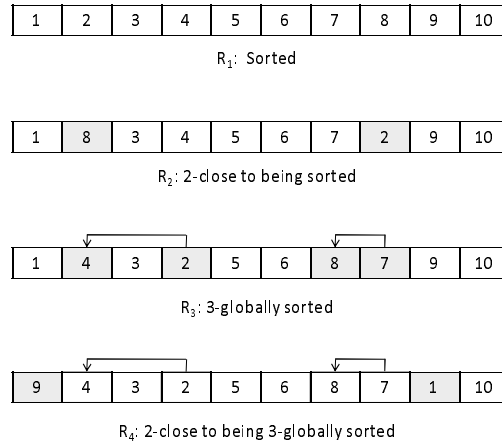


Figure 2.1: Sorted and nearly-sorted relations

**Example 3.** In Figure 2.1 we illustrate the different definitions of nearly-sorted relations. In the sketched relations, the cells contain numbers that refer to the keys of the tuples. Relation  $R_1$  is sorted. Relation  $R_2$  is 2-close to being sorted, because removing 8 and 2 makes it a sorted relation. Relation  $R_3$  is 3-globally sorted, because every two tuples that the difference between their locations is at least 3 comply with the order of the keys. Finally, the relation  $R_4$  is (2, 3)-nearly sorted, because outside the tuples whose keys are 9 and 1, we actually have a part of  $R_3$ .

In the sequel we show that nearly sorted relations indeed admit an efficient algorithm for sorting or computing other operators. But as mentioned earlier, we also need a way to efficiently decide whether to apply an algorithm tailored for the nearly sorted case. For this we use a *property tester*. Rather than provide the standard testing theory definition, we provide here a specific definition for our application; we also change the terminology accordingly (e.g., “probe complexity” here is what testing theory refers to as “query complexity”)

**Definition 2.7.** Given four integers  $k' \leq k$  and  $\ell' \leq \ell$ , a  $([k', k], [\ell', \ell])$ -sortedness test with probe complexity  $q$  and error probability  $\delta$  is an algorithm making at most  $q$  read operations from the relation (all of them using random access), and satisfying the following.

- If the relation is  $(k', \ell')$ -nearly sorted, then the algorithm outputs ACCEPT with probability at least  $1 - \delta$ .
- If the relation is not even  $(k, \ell)$ -nearly sorted, then the algorithm outputs REJECT with probability at least  $1 - \delta$ .

Note that in 2.7 the parameters may (and will) depend on the size of the relation,  $n$ . The parameter  $\delta$  will usually be held to the constant  $1/3$ , but making  $\delta$  smaller with a logarithmic in  $1/\delta$  penalty on  $q$  will turn out to be an easy amplification procedure.

The above definition in fact deviates somewhat from the standard testing definition. In the standard theory an  $\epsilon$ -test for being  $\ell$ -globally sorted would correspond to what we defined here as a  $([0, \epsilon n], [\ell, \ell])$ -sortedness test. Having a  $k' > 0$  lower bound makes it relate to the stronger notion of a *tolerant test*, and having two instead of one  $\ell$  parameter (as we shall see below) will allow us to improve the efficiency of the test considerably, in fact making its probe complexity independent of  $\ell$ .

## Chapter 3

# Outline of the suggested query evaluation strategy

As mentioned in the introduction, for some useful (small enough with respect to the computer memory) parameters  $k_0$  and  $\ell_0$  a typical relation  $R$  is likely to be  $(k_0, \ell_0)$ -nearly sorted. Based on this, our first ingredient is an efficient “correction” algorithm – Algorithm 2 below, that for any two integers  $k, \ell$  can evaluate the sort operation on a  $(k, \ell)$ -nearly sorted relation more efficiently than a conventional algorithm given sufficient memory. We use the sort operation itself as an example because it is relatively simple to describe, while later we explain how other operations will work by having basically the same sorting procedures plugged in. The algorithm is also capable of reporting whether the evaluation succeeded or not, so we know whether a fall-back procedure is needed.

The second ingredient in our strategy is an extremely efficient probabilistic test (Algorithm 3) that for any integers  $k, \ell$  can distinguish between the case where  $R$  is  $(k, \ell)$ -nearly sorted (“Case Y”) and the case where  $R$  is not even  $(6k, 6\ell)$ -nearly sorted (“Case N”).

The exact details on the resources that these algorithms require are given in the relevant chapters, but for the discussion here we should think of the probabilistic test as the cheapest algorithm, and of the correction algorithm as significantly cheaper than a conventional evaluation algorithm.

### Suggested strategy:

1. First we apply the test with parameters  $k_0, \ell_0$ .
2. If the prediction of the test is “Case Y”:
  - (a) We apply the correction algorithm with parameters  $6k_0$  and  $6\ell_0$ . If the correction algorithm succeeds, then we are done.

- (b) Otherwise, if we were using Algorithm 2 then we can continue running its special fall-back mode (see Remark 4.4 below) or decide to start from scratch with a conventional algorithm, depending on how soon it failed. If we were using the simpler Algorithm 1 then the fall-back is easier – we can just continue with a conventional algorithm from where it failed.
3. If the test predicted “Case N”, then we use a conventional evaluation algorithm without attempting the correction algorithm at all.

By sufficiently amplifying the success probability of our test, reaching Step 2b will be rare enough so that the average cost of falling back to the fall-back mode of our algorithm (rather than using a conventional algorithm from the beginning) will be negligible. The fall-back overhead, while best avoided, is generally within the same order of magnitude as that of running the algorithm itself. Also, reaching Step 3 with relations that could have benefited much by our algorithm (again through an error of the testing procedure) will have a negligible average cost. For relations reaching Step 3 that in fact satisfy Case N we only have the additional cost of performing the test (when compared to a query processor that always performs a conventional algorithm); a preliminary experiment (see Section 7.6) suggests that the percentage of those would typically be small enough to have an overall average net gain by using our procedure.

Choosing the right candidate parameters  $k_0, \ell_0$  can be done empirically, based on past statistics, but this is not necessary. In Section 7.1 we show a probabilistic algorithm that with a slight overhead over the original tolerant tester can output a comprehensive set of candidate pairs  $(k_i, \ell_i)$  (up-to specified precision) for which the given relation  $R$  falls under Case Y.

In settings where the relations change slowly over time we can also cache the results of the test for use in future queries, rather than test the relation every time.

## Chapter 4

# Exploiting Near Sortedness

In this chapter we first present Algorithm 1, which sorts  $(k, \ell)$ -nearly sorted relations in two sequential passes. In the first pass Algorithm 1 acts similarly to the well known Replacement-Selection algorithm (see [6])<sup>1</sup>, while it also collects a set of misplaced entries (*i.e.*, a set of at most  $k$  tuples that must be removed to make the relation  $\ell$ -globally sorted). In the second pass, the collected tuples are distributed to their final positions. Note that although we have a fail state in the description of Algorithm 1, we later discuss a fall-back mechanism that preserves most of the work already done.

Albeit sometimes useful in itself, the main function of this algorithm is to serve as a warm-up for the more efficient Algorithm 2, whose analysis is based on the analysis of Algorithm 1. The main change in Algorithm 2 is that it defers all writing to the second pass, managing to do away with the writing of an intermediate file. Its fall-back mode is on the other hand less efficient than that of Algorithm 1, so a property testing procedure for being  $(k, \ell)$ -nearly sorted is more important there.

**Theorem 4.1.** *If  $R$  is a  $(k, \ell)$ -nearly sorted relation, then Algorithm 1 does not reach its fail state and the resulting relation  $OUT$  is strictly sorted. Furthermore, Algorithm 1 makes only two sequential passes on  $R$ , uses memory of size  $O(k + \ell)$ , and makes  $O(n \cdot \log(\ell + k))$  computing operations.*

*Proof.* The fact that only two passes are made as well as the bounds on memory size and the number of computing operations are clear from the description of Algorithm 1. Now we prove that the algorithm sorts any relation which is  $(k, \ell)$ -nearly sorted.

It is easy to see that if the algorithm did not fail in the first pass, then the intermediate result (that is written in  $TMP$ ) is sorted. If this is the case, then the second pass is just the standard merge between two sorted lists, and hence the resulting relation (written

---

<sup>1</sup>In fact, our algorithm starts out identically to the Replacement-Selection algorithm, but the analysis given here shows that, specifically for nearly sorted relations, much stronger properties hold.

---

**Algorithm 1** (Sorts a  $(k, \ell)$ -nearly sorted relation  $R$ .)

---

```

create two binary heaps  $S, G$ 
insert the first  $k + \ell + 1$  tuples  $(R[1], \dots, R[k + \ell + 1])$  into  $S$ 
 $i_{write} \leftarrow 1$ 
for  $i_{read} = |S| + 1$  to  $n$  do {first pass}
  if  $S = \emptyset$  then
    FAIL
  end if
   $last\_written \leftarrow \min\{x \in S\}$ 
  write  $last\_written$  to  $TMP[i_{write}]$ 
   $S \leftarrow (S \setminus \{last\_written\})$ 
   $i_{write} \leftarrow i_{write} + 1$ 
  if  $R[i_{read}] \geq last\_written$  then
    insert  $R[i_{read}]$  into  $S$ 
  else
    insert  $R[i_{read}]$  into  $G$ 
  end if
end for
append all tuples in  $S$  to  $TMP$ , in sorted order
 $i_{write} \leftarrow 1$ 
for  $i_{read} = 1$  to  $n - |G|$  do {second pass}
   $x \leftarrow \min\{y \in G\}$ 
  if  $x > TMP[i_{read}]$  then
    write  $TMP[i_{read}]$  to  $OUT[i_{write}]$ 
  else
    write  $x$  to  $OUT[i_{write}]$ 
     $G \leftarrow (G \setminus \{x\}) \cup \{TMP[i_{read}]\}$ 
  end if
   $i_{write} \leftarrow i_{write} + 1$ 
end for
append all tuples in  $G$  to  $OUT$ , in sorted order

```

---

to  $OUT$ ) will be fully sorted. So, we only need to prove that if  $R$  is  $(k, \ell)$ -nearly sorted, then Algorithm 1 cannot fail. Observe that in every stage of the first pass  $|S| + |G|$  equals  $k + \ell + 1$ , and therefore showing that  $|G|$  never exceeds  $k$  implies that  $S$  stays nonempty, preventing the algorithm from failing.

Let  $\mathcal{E}(R)$  denote the collection of subsets  $E \subseteq [n]$  of at most  $k$  indices, such that for every  $E \in \mathcal{E}(R)$ , if we restrict  $R$  to indices  $[n] \setminus E$  then we get an  $\ell$ -globally sorted relation.

Let in addition  $D = \{j \in [n] : \forall E \in \mathcal{E}(R), j \in E\}$  denote the set of indices that *must* be removed from  $R$  in order to make it  $\ell$ -globally sorted by at most  $k$  removals. Observe that  $|D| \leq k$ . With a slight abuse of notation, let us also denote by  $D$  the set of tuples that appear at indices  $D$  of the relation  $R$ .

We claim that at every stage of the first pass,  $G \subseteq D$  and hence  $|G| \leq k$ . To see

this, notice that whenever a tuple at index  $i$  is inserted into  $G$ , it is strictly smaller than  $last\_written$ . Using this observation, we prove the claim  $G \subseteq D$  by induction on  $i$ .

For  $i \leq k + \ell + 1$  the claim trivially holds (since these indices are unconditionally inserted into  $S$ ). Now let  $i > k + \ell + 1$ . By the induction hypothesis, before treating  $R[i]$  we had  $G \subseteq D$ . If  $R[i] \geq last\_written$  then  $G$  remains the same and we are done. Otherwise, since  $R[i] < last\_written$  there are at least  $|S| = k + \ell + 1 - |G| \geq \ell + 1$  tuples in  $S$  that are strictly larger than  $R[i]$ . All these tuples originally appeared before  $R[i]$ . Consequently, at least  $|S| - \ell \geq 1$  of these tuples appeared in indices lower than  $i - \ell$ . Assume that  $i \notin D$  and let  $E \in \mathcal{E}(R)$  be such that  $i \notin E$ . Then all the corresponding  $|S| - \ell \geq 1$  indices should be in  $E$ , because they form a violation of  $\ell$ -global sortedness together with  $i$ . By the induction hypothesis,  $G \subseteq D \subseteq E$  and hence  $|E| \geq |G| + |S| - \ell \geq k + 1$ , contradicting the fact that  $|E| \leq k$ . Hence  $i$  must be in  $D$ , concluding the proof.  $\square$

**Remark 4.2.** *In an actual implementation we can (and should) replace the failure mode in Algorithm 1 with a fall-back to a traditional sorting algorithm. For example, instead of failing, the algorithm can just reset  $last\_written$  and start writing another run (monotone subsequence), repeating this as many times as is necessary. In other words, we just fall-back to the Replacement Selection algorithm for creating runs that are as long as possible. Next, instead of moving to the second pass, a traditional merge-sort can be performed.*

*Implementing a fall-back is necessary in part because of the probabilistic nature of the testing algorithms. In Section 7.2 below we touch upon the expected amount of overhead when the  $k$  parameter in the near-sortedness of the input is somewhat larger than available memory, bounding it when the error in the estimation of  $k$  is not too large (which will usually be the case).*

Now we present a sorting algorithm which is an improvement over Algorithm 1 as it saves significantly on write operations, and in fact writes nothing apart from the output itself. Apart from the clear saving in write operations in itself, not writing any intermediate result to disk makes the algorithm easier to combine with other database operations, because its output can be piped directly to the algorithm processing the next operation. The only disadvantage in the improved algorithm is that sometimes not all of the work already done upon reaching a failure mode is recoverable, so it is all the more important to use the test of Chapter 6 first. The analysis of the improved algorithm is based on the analysis of Algorithm 1.

**Theorem 4.3.** *If  $R$  is a  $(k, \ell)$ -nearly sorted relation, then Algorithm 2 does not reach its fail states and the resulting relation  $OUT$  is strictly sorted. Furthermore, Algorithm 2 makes only two sequential passes on  $R$ , uses memory of size  $O(k + \ell)$ , and makes*

$O(n \cdot \log(k + \ell))$  computing operations; also, it never writes any intermediate file, only the sorted output.

---

**Algorithm 2** (Improved sort for a  $(k, \ell)$ -nearly sorted relation  $R$ .)

---

```

create two binary heaps  $S, G$ 
insert the first  $k + \ell + 1$  tuples  $(R[1], \dots, R[k + \ell + 1])$  into  $S$ 
 $last\_handled \leftarrow MIN\_VAL$ 
for  $i_{read} = |S| + 1$  to  $n$  do {first pass}
  if  $S = \emptyset$  then
    FAIL
  end if
   $last\_handled \leftarrow \min\{x \in S\}$ 
   $S \leftarrow (S \setminus \{last\_handled\})$ 
  if  $R[i_{read}] \geq last\_handled$  then
    insert  $R[i_{read}]$  into  $S$ 
  else
    insert  $R[i_{read}]$  into  $G$ 
  end if
end for
empty  $S$ 
let  $G[1], \dots, G[|G|]$  be the sorted order of  $G$ 's elements
insert the first  $k + \ell + 1$  tuples  $(R[1], \dots, R[k + \ell + 1])$  into  $S$ 
 $i_{write} \leftarrow 1, i_g \leftarrow 1$ 
 $last\_handled \leftarrow MIN\_VAL$ 
for  $i_{read} = |S| + 1$  to  $n$  do {second pass}
  if  $S = \emptyset$  then
    FAIL
  end if
   $last\_handled \leftarrow \min\{x \in S\}$ 
  while  $i_g \leq |G|$  and  $G[i_g] \leq last\_handled$  do
    write  $G[i_g]$  to  $OUT[i_{write}]$ 
     $i_g \leftarrow i_g + 1, i_{write} \leftarrow i_{write} + 1$ 
  end while
  write  $last\_handled$  to  $OUT[i_{write}]$ 
   $i_{write} \leftarrow i_{write} + 1$ 
   $S \leftarrow (S \setminus \{last\_handled\})$ 
  if  $R[i_{read}] \geq last\_handled$  then
    insert  $R[i_{read}]$  into  $S$ 
  end if
end for
append  $\{G[i_g], \dots, G[|G|]\} \cup S$  to  $OUT$ , in sorted order

```

---

*Proof.* The fact that only two passes are made as well as the bounds on memory size and the number of computing operations are clear from the description of Algorithm 2. To prove that the algorithm sorts any relation which is  $(k, \ell)$ -nearly sorted, we use the proof of Theorem 4.1.



We note that both passes of our algorithm in fact mimic the first pass of Algorithm 1, so that the analysis there still holds (in fact unless a fall-back was implemented for the first fail state, our algorithm will never reach the second fail state, whether the input is  $(k, \ell)$ -nearly sorted or not). The first pass of our algorithm is identical to the first pass of Algorithm 1, with the only difference being that nothing is written to disk. In the end we are left with  $G$ , which holds up to  $k$  tuples of the relation.

The second pass again generally follows the first pass of Algorithm 1, only here tuples are not inserted to  $G$ , because already in the beginning  $G$  contains all tuples that would have been passed to it. Additionally, this pass follows the writing pattern of the first pass of Algorithm 1, only here we merge the (already known in advance) content of  $G$  into the output stream. Therefore this essentially combines the writing actions of the two passes of Algorithm 1, resulting in a fully sorted output.  $\square$

**Remark 4.4.** *In the case where algorithm reaches the fail state in the first pass, the natural instinct is to continue with algorithm by writing the contents of  $G$  to a temporary file on the disk, clearing it, and then populating  $S$  with  $k + \ell + 1$  new tuples from the input (without resetting `last_handled`). However, this will not work – while out of place tuples that come “too late” (i.e. well after their position according to the ordered relation) will not pose a problem here, tuples that come “too early” may set the value of `last_handled` so high that the rest of the relation will land in  $G$ .*

*There is still a fall back procedure that is more beneficial than just restarting the sort using a different algorithm when the error in our estimation of  $k$  is not too large (e.g., whenever the true  $k$  still satisfies  $k = o(n)$ ). See Section 7.3.*

## Chapter 5

# Evaluating Other Operators

In this chapter we discuss efficient evaluation of some order-preferring operations over nearly-sorted relations. For all operations here we use at our core Algorithm 2, which is the one more suited for integration into larger operation. For the purpose here we do not describe the fall-back procedures in case we reach a fail state, as these would be the expected ones. We outline the algorithms without formal details whenever these follow from standard procedures in relational query evaluation.

**Intersection and other set operations.** Consider the computation of an intersection of two relations,  $R_1$  that is  $(k_1, \ell_1)$ -nearly sorted and  $R_2$  that is  $(k_2, \ell_2)$ -nearly sorted, where for both relations the entire schema is the sort key in some order. Suppose that the memory is large enough to hold  $2k_1 + \ell_1$  tuples of  $R_1$ ,  $2k_2 + \ell_2$  tuples of  $R_2$ , and buffers for the input and output.

In the first stage, we just perform the first pass of Algorithm 2 on  $R_1$  and  $R_2$  (in some cases, for example if they are on different disks, this is best done in parallel), and obtain the corresponding heaps  $G_1$  for  $R_1$  and  $G_2$  for  $R_2$ . Recall that the first pass of Algorithm 2 does not produce any output.

Then we go in parallel over  $R_1$  and  $R_2$  again, and perform a procedure similar to the second pass of the sorting algorithm on each of them, but with the following change: Instead of writing the sorted output to disk, we instead pipe it to the algorithm that performs intersection using the merge procedure for two sorted relations. In fact, we do not keep running the second pass of the sorting algorithm over  $R_1$  and  $R_2$  unconditionally, but use it as an iterator – we buffer sorted subsequences of  $R_1$  and  $R_2$ , and whenever a buffer is exhausted by the merge-intersection procedure we run more iterations of the loop in the sorting algorithm for the corresponding relation so as to fill the buffer again.

Accommodating bag (multiset) intersection is an easy extension of the above procedure, one just needs to keep track also of the original locations in the files of the tuples that are stored in  $G_1$  and  $G_2$ . If we were using Algorithm 1 instead, we would have

needed to run over both the result of the intersection from first pass and the original relations to make sure that we got the correct number of duplicate entries.

Using the above idea (with Algorithm 2) for set union, or set or bag difference, works in much the same way: running sorting algorithm instances in parallel on  $R_1$  and  $R_2$  and piping the output of the second stage to the corresponding merging algorithm. Bag union by itself is not an order preferring operation, but if it is part of a larger expression involving order preferring operations then it may still be better to combine it with the sorting procedure so that its output would be sorted.

**Natural join (equijoin).** The computation of a natural join is performed similarly to the computation of intersection, but here there is a new issue that many tuples can agree on the value of the common attributes in the join. Suppose that at most  $m_1$  tuples of  $R_1$  can agree at one time on the values of the attributes common to  $R_1$  and  $R_2$ , while at most  $m_2$  tuples of  $R_2$  can agree on them. In this case for the merging join algorithm during the second stage to work, we would need a memory big enough to hold a total of  $(2k_1 + \ell_1)$  tuples of  $R_1$ ,  $(2k_2 + \ell_2)$  tuples of  $R_2$ , and additionally either  $m_1$  tuples of  $R_1$  or  $m_2$  tuples of  $R_2$  (as well as sufficient buffers). If this does not hold, then we would need to accommodate for saving and retrieving the state of the second stage sorting algorithm over (say)  $R_2$ . Then, given a large subsequence of tuples from  $R_2$  that agree on a common attribute, we can save the state at the beginning of the sequence, and reset the algorithm to this state for every tuple from  $R_1$  that needs to be joined with that subsequence.

## Chapter 6

# Testing for Near Sortedness

In this chapter we develop a tolerant sortedness test, namely a  $([k, 6k], [\ell, 6\ell])$ -sortedness test, and prove its correctness. The reason for the test to be tolerant (*i.e.*, use  $[k, 6k]$  rather than  $[0, k]$ ) is so that more instances for which Algorithm 2 (or Algorithm 1) still works are accepted by the test.

Some inaccuracy in the  $k$  parameter cannot be avoided (*i.e.*, there is no test with  $[k, k]$  parameters), though currently we do not know how much lower than a factor of 6 one can go. As for the  $\ell$  parameter, while there is a test that is fully accurate in  $\ell$ , its number of queries depends badly on  $\ell$ . In contrast, the test in this chapter has no dependency on  $\ell$  at all. The Appendix contains descriptions and proofs for simpler (non-tolerant) tests, and it can be read as a warm-up to this chapter.

The test presented here is non-adaptive, meaning that it can decide which tuples to read before the first reading of a tuple – only the final decision to accept or reject depends on the actual values read. This can serve to reduce the overhead further: Instead of reading the queries in the order that they are used in the test (which requires fully random-access reads), we can first decide the queries and then make them in the order of their positions in the file. Moreover, modern operating systems can first receive the entire list of all reads to be made and then optimize their order of execution further for the file system involved.

**Theorem 6.1.** *Algorithm 3 is a  $([k, 6k], [\ell, 6\ell])$ -sortedness test that makes  $O(\frac{n}{k} \log \frac{n}{k} \log n \log \log n)$  probes and errs with probability at most  $1/3$ .*

Notice that the probe complexity is independent of  $\ell$ . In property testing,  $k$  is usually set to  $\epsilon n$  for some small constant  $\epsilon$ . In these terms, the probe complexity of our test is nearly logarithmic in  $n$ , which is known to be optimal even for non-tolerant simple monotonicity testing [2].

In the description of Algorithm 3 we use an undefined parameter  $s$ , the value of

---

**Algorithm 3** ( $([k, 6k], [\ell, 6\ell])$ -sortedness test)

---

```

 $a \leftarrow 0$ 
for  $j = 1$  to  $s$  do
  pick  $i_j \in [n]$  uniformly at random
  call Algorithm 4 for  $i_j$  with confidence parameter  $\delta = \frac{1}{6s}$ 
  if  $i_j$  is reported to be active then
     $a \leftarrow a + 1$ 
  end if
end for
if  $a \leq s \frac{5.5k}{n}$  then
  return ACCEPT
else
  return REJECT
end if

```

---

which will be set later in the proof of Theorem 6.1, and call Algorithm 4 that we define subsequently. It will be clear from the proof that the success probability can be easily amplified by increasing  $s$ . First we need the following definition and lemmas.

In the following we will say that a pair  $(i, j)$  violates  $\ell$ -global sortedness if  $i \leq j - \ell$  and  $R[i] > R[j]$ .

**Definition 6.2.** Let  $(i, j)$  be a pair of indices with  $i < j$  that violate the  $\ell$ -global sortedness of  $R$ . We say that  $i$  is  $(\delta, \ell)$ -active with  $j$  (for  $\delta > 0$ ) if at least a  $\delta$ -fraction of the indices in  $[i + 1, j]$  violate  $\ell$ -global sortedness together with  $i$ . Similarly, we say that  $j$  is  $(\delta, \ell)$ -active with  $i$  if at least a  $\delta$ -fraction of the indices in  $[i, j - 1]$  violate  $\ell$ -global sortedness together with  $j$ . We say that an index  $i$  is simply  $(\delta, \ell)$ -active if it is  $(\delta, \ell)$ -active with some index  $j \in [n] \setminus \{i\}$ .

**Lemma 6.3.** Let  $R$  be a relation and let  $k, \ell$  be two positive integers. For every  $\delta \in (0, 1/2]$  let  $d(\delta)$  denote the number of  $(\delta, \ell)$ -active indices in  $R$ .

- if  $R$  is  $(k, \ell)$ -nearly sorted then  $d(\delta) \leq k + k/\delta$ , and in particular  $d(1/4) \leq 5k$ ;
- if  $R$  is not  $(6k, 6\ell)$ -nearly sorted then  $d(1/3) \geq 6k$ .

*Proof.* The proof of second part of the lemma is standard in monotonicity testing: If  $(i, j)$  is a pair violating  $6\ell$ -global sortedness, then every  $k$  between  $i + \ell$  and  $j - \ell$  violates  $\ell$ -global sortedness with either  $i$  or  $j$ . Hence, at least one of  $i$  or  $j$  violates  $\ell$ -global sortedness with at least  $\frac{j-i-2\ell}{2} \geq \frac{j-i}{3}$  indices in the interval, making it  $\frac{1}{3}$ -active. As the set of all  $\frac{1}{3}$ -active indices now intersects all pairs violating  $6\ell$ -global-sortedness, its size must be at least the distance  $6k$ .

Assuming that  $R$  is  $(k, \ell)$ -nearly sorted, we now prove the first part of the lemma. For this we will use some methods from [1], together with additional arguments that are

specific to globally sorted relations. Let  $E \subseteq [n]$  be a set of at most  $k$  indices, whose dismissal makes  $R$   $\ell$ -globally sorted. Such a set must exist since we assumed that  $R$  is  $(k, \ell)$ -nearly sorted. By Lemma 2.6 we can fix new values for these indices so that the resulting relation is  $\ell$ -globally sorted. From now on let us fix a set  $E$  as above, and a sequence of new values for the corresponding tuples as per Lemma 2.6.

Next we are going to label some of the indices of  $R$ . For every  $i \in E$ , we label  $i$  as *high* if its tuple should be decreased (replaced by one with a lower key value under the above correction), and we label it as *low* otherwise. For each  $(\delta, \ell)$ -active index  $i \in [n] \setminus E$  (we stress that  $i$  is *not* in  $E$ ) we assign some index  $j_i$  that witnesses the fact that  $i$  is  $(\delta, \ell)$ -active. If  $j_i > i$ , then the label of  $i$  is *big*; otherwise its label is *small*. Notice that each index can have at most one label as above (every  $i$  is either high, low, big, small or has no label at all). Our aim is to bound the number of indices that are  $(\delta, \ell)$ -active, which is upper bounded by the number of labeled indices. By definition, the number of high and low indices is at most  $k$ , so it is enough to show that the number of big and small indices is at most  $k/\delta$ . By letting  $k_{low}$  and  $k_{high}$  denote the number of low and high indices (respectively), we show how to bound the number of big indices by  $k_{low}/\delta$ . An analogous argument works for bounding the number of small indices by  $k_{high}/\delta$ .

We start by assigning weight 1 to every big index. Then, for each big index  $i$ , in decreasing order, we divide the weight of  $i$  among all the low indices  $h$  such that  $i \leq h \leq j_i$  and  $R(h) < R(i)$ . We “spread” the weight of  $i$  in a way that maximizes the minimal weight of the receiving indices (the  $h$ 's). Our goal is to show that after this process, no low index has weight more than  $1/\delta$ , and hence the total initial weight of the big indices (which is exactly equal to their amount) was at most  $k_{low}/\delta$  as required.

Suppose on the contrary that this is not the case, so that some low index  $g$  got weight  $(1 + \epsilon)/\delta$  for some  $\epsilon > 0$ . Let  $i$  be the first (in reverse order) big index that caused  $g$  to reach weight  $(1 + \epsilon)/\delta$ . By definition, this event happened while the weight of  $i$  was spread among the low indices  $h$  such that  $i < h \leq j_i$  and  $R(h) < R(i)$ . Denote their number by  $b$ . From the way the weight is spread, *all* of these low indices must have weight at least  $(1 + \epsilon)/\delta$ . Hence their total weight is at least  $b((1 + \epsilon)/\delta)$ . By the definition of  $(\delta, \ell)$ -active indices,  $b \geq \delta(j_i - i + 1)$ , so their total weight is at least  $(1 + \epsilon)(j_i - i + 1)$ . Since we iterate on the  $i$ 's in decreasing order, none of these  $h$ 's could gain any weight before step  $j_i$ , and therefore we should have  $(1 + \epsilon)(j_i - i + 1) \leq j_i - i + 1$ , which is a contradiction.  $\square$

The next lemma allows us to distinguish between indices that are  $(1/3, \ell)$ -active and indices that are not even  $(1/4, \ell)$ -active.

**Lemma 6.4.** *Algorithm 4, given an index  $i \in [n]$  and confidence parameter  $\delta > 0$ ,*

satisfies the following:

- if  $i$  is  $(1/3, \ell)$ -active, it outputs ACTIVE with probability at least  $1 - \delta$ ;
- if  $i$  is not even  $(1/4, \ell)$ -active, it outputs INACTIVE with probability at least  $1 - \delta$ ;
- its probe complexity is  $O(\log \frac{1}{\delta} \log n \log \log n)$ .

The constants  $\alpha > 0$  and  $t \in \mathbb{N}$  in the definition of Algorithm 4 are set later in the proof.

---

**Algorithm 4** (tests if  $i$  is  $(1/3, \ell)$ -active or not even  $(1/4, \ell)$ -active)

---

```

for  $h = \lceil \log_{1+\alpha}(\ell + 1) \rceil$  to  $\lceil \log_{1+\alpha} n \rceil$  do
   $left \leftarrow 0, right \leftarrow 0$ 
  for  $j = 1$  to  $a = t \log \frac{1}{\delta} \log \log n$  do
    pick  $i_j \in [\ell + 1, (1 + \alpha)^h]$  uniformly at random
    if  $R[i] > R[i + i_j]$  then
       $right \leftarrow right + 1$ 
    end if
    if  $R[i] < R[i - i_j]$  then
       $left \leftarrow left + 1$ 
    end if
  end for
  if  $right > \frac{2}{7}a$  or  $left > \frac{2}{7}a$  then
    return ACTIVE
  end if
end for
return INACTIVE

```

---

*Proof.* Assume first that  $i$  is  $(1/3, \ell)$ -active, and let  $j_i$  be an index that witnesses this fact, so there exist at least  $\frac{j_i - i}{3}$  indices lying between  $i$  and  $j_i$  that violate  $\ell$ -global sortedness with  $i$ . We assume without loss of generality that  $j_i > i$ . Let  $h_0 \in \mathbb{N}$  be such that  $(1 + \alpha)^{h_0} \leq j_i - i \leq (1 + \alpha)^{h_0 + 1}$ . Then in the interval  $[i, i + (1 + \alpha)^{h_0 + 1}]$  at least a  $1/3 - \alpha$  fraction of the indices violate  $\ell$ -global sortedness with  $i$ . We fix  $\alpha$  to be small enough (say  $1/100$ ), so that  $1/3 - \alpha$  is much closer to  $1/3$  than to  $2/7$ . For large enough  $t$  (matching the parameters in Chernoff bounds), after the  $h_0 + 1$ 'th iteration of the internal loop, with probability at least  $1 - \delta$  the value of  $right$  will be sufficiently close to  $(1/3 - \alpha)(1 + \alpha)^{h_0 + 1}$ . In particular the value of  $right$  will exceed  $\frac{2}{7}(1 + \alpha)^{h_0 + 1}$ , hence the outcome will be ACTIVE as required for the first part of the lemma.

To prove the second part of the lemma, we use a similar argument. Namely, if the index  $i$  is not even  $(1/4, \ell)$ -active, then for all  $h$  the fraction of violating (with respect to  $i$ ) indices between  $i$  and  $i + (1 + \alpha)^h$  (and similarly between  $i$  and  $i - (1 + \alpha)^h$ ) is at most  $1/4$ . But now we must make sure that no error occurred, meaning that the values

of the counters *right* and *left* did not deviate too much in *any* of the  $O(\log n)$  iterations of the outer loop. We can solve this problem by amplifying the success probability to  $1 - \Omega(1/\log n)$ . This is the reason that we have the extra  $\log \log n$  factor in the number of iterations of the internal loop.  $\square$

*Proof of Theorem 6.1.* First notice that the confidence parameter  $\delta$  in the executions of Algorithm 4 is set to  $\frac{1}{6s}$ , so that with probability at least  $1 - 1/6$  Algorithm 4 did not err during any of the  $s$  executions.

If  $R$  is  $(k, \ell)$ -nearly sorted, then according to Lemma 6.3 (first item), the number of  $(1/4, \ell)$ -active indices in  $R$  is at most  $5k$ . Conditioned over the event that none of the executions of Algorithm 4 err (recall that this event occurs with probability at least  $5/6$ ) we have that the expected value of  $a$  is at most  $s \frac{5k}{n}$ . The probability that Algorithm 3 returns REJECT in this case is equal to the probability that the random variable  $a$  (being a sum of  $s$  independent random variables) deviates from its expectation by a multiplicative factor of 0.1. This probability can be bounded by  $1/6$  by taking  $s = O(n/k)$ , so altogether Algorithm 3 returns ACCEPT with probability at least  $2/3$  as required.

If  $R$  is  $6k$ -far from being  $6\ell$ -globally sorted, then according to Lemma 6.3 (second item), the number of  $(1/3, \ell)$ -active indices in  $R$  is at least  $6k$ . Conditioned over the event that none of the executions of Algorithm 4 err, the expected value of  $a$  is at least  $s \frac{6k}{n}$ . So similarly to the previous case, the probability that the random variable  $a$  deviates from its expectation by a multiplicative factor of 0.08 can be bounded by  $1/6$ , and altogether Algorithm 3 returns REJECT with probability at least  $2/3$  as required. The probe complexity of Algorithm 3 is  $s \cdot O(\log \frac{1}{1/s} \log n \log \log n) = O(\frac{n}{k} \log \frac{n}{k} \log n \log \log n)$ .  $\square$



# Chapter 7

## Discussion

In the following we touch upon some possible extensions of our methods for detecting and exploiting near-sortedness, and related issues.

### 7.1 Testing for several values of $k$ and $\ell$ at once

In all of previous chapters we used algorithms that take the values of  $k$  and  $\ell$  in advance. However, we may be interested in learning actual approximate values of  $k$  for many values of  $\ell$  at once. First, our computer memory puts constraints only on  $k + \ell$  (for Algorithm 1) or  $2k + \ell$  (For Algorithm 2) and we may want to search for an optimal  $\ell$  for which this fits our memory (every input  $R$  has for every  $\ell$  a minimum  $k$  for which it is  $(k, \ell)$ -nearly sorted, the worst case being  $k = n - 1$ ). Second, sometimes we would like to use our sorting algorithm even if we know that it may fail (this scenario fits Algorithm 1), because it could still lead to faster sorting (see Section 7.2 and Section 7.3 below).

There is an easy extension of Algorithm 3 that allows to efficiently test for many values of  $k$  and  $\ell$  at once. First, we construct Algorithm 5, a version of Algorithm 4 which tests whether an element is active for every possible  $\ell = c^r$ , where  $c > 1$  is any fixed constant (that determines the number of different  $\ell$ 's we inspect) and  $r = 1, 2, \dots, \log_c n$ . We restrict ourselves to powers of  $c$  so that this output would be of manageable size, and this would still give a good approximation of the optimal  $k$  and  $\ell$ . The parameters  $t$  and  $\alpha$  used below are the same as in Algorithm 4.

Now we can use Algorithm 6, a variant of Algorithm 3. It uses the same  $s$  as Algorithm 3, but here it is calculated not as a function of  $k$  (which is not provided in advance) but as a function of a desired approximation parameter  $\hat{k}$ , which should be set equal to a small constant fraction of the available computer memory. Here we also keep count of every possible  $\ell = c^r$ . In addition, instead of deciding on ACCEPT or REJECT, we just output all counters after the appropriate normalization.

---

**Algorithm 5** (tests for every  $\ell = c^r$  if  $i$  is  $(1/3, \ell)$ -active or not even  $(1/4, \ell)$ -active)
 

---

```

 $A[0], A[1], \dots, A[\lceil \log_c n \rceil] \leftarrow \text{INACTIVE}$ 
for  $h = 1$  to  $\lceil \log_{1+\alpha} n \rceil$  do
   $LF[0], LF[1], \dots, LF[\lceil \log_c n \rceil] \leftarrow 0$ 
   $RT[0], RT[1], \dots, RT[\lceil \log_c n \rceil] \leftarrow 0$ 
  for  $j = 1$  to  $a = 2t \log_{\frac{1}{\delta}} \log \log n$  do
    pick  $i_j \in [1, (1 + \alpha)^h]$  uniformly at random
    if  $R[i] > R[i + i_j]$  then
      for  $r = 0$  to  $\lfloor \log i_j \rfloor$  do
         $RT[r] \leftarrow RT[r] + 1$ 
      end for
    end if
    if  $R[i] < R[i - i_j]$  then
      for  $r = 0$  to  $\lfloor \log i_j \rfloor$  do
         $LF[r] \leftarrow LF[r] + 1$ 
      end for
    end if
  end for
  for  $r = 0$  to  $\lceil \log_c n \rceil$  do
    if  $RT[r] > \frac{2}{7}a$  or  $LF[r] > \frac{2}{7}a$  then
       $A[r] \leftarrow \text{ACTIVE}$ 
    end if
  end for
end for
return  $A$ 

```

---

The running time of Algorithm 6 is  $\tilde{O}(n/\hat{k})$ , and with probability at least  $\frac{2}{3}$  it provides for every  $\ell = c^r$  an approximation  $k$  such that  $R$  is  $(6k + \hat{k}, 6\ell)$ -nearly sorted while not being  $(k - \hat{k}, \ell)$ -nearly sorted. The proof is a straightforward extension of the argument given in Chapter 6, since having the confidence parameter  $(1 - \delta)$  amplified to  $(1 - 1/\text{polylog}(n))$  allows us to simply apply a union bound over all values  $\ell = c^r$ .

---

**Algorithm 6** (approximates  $k = k(\ell)$  for every  $\ell = c^r$ )
 

---

```

 $B[0], B[1], \dots, B[\lceil \log_c n \rceil] \leftarrow 0$ 
for  $j = 1$  to  $b = s \lceil \log n \rceil = O(n \log n / \hat{k})$  do
  pick  $i_j \in [n]$  uniformly at random
  call Algorithm 4 with index  $i_j$  and  $\delta = \frac{1}{6b}$  to obtain  $A$ 
   $B \leftarrow B + A$  (coordinate-wise)
end for
return  $\frac{n}{b}B$  (coordinate-wise)

```

---

## 7.2 Distances larger than memory

Suppose that there is enough memory for handling a  $(k, \ell)$ -nearly sorted relation, but we attempt to use Algorithm 1 when the input is in fact not  $(k, \ell)$ -nearly sorted. If we implement Remark 4.2 then we will fall-back to the Replacement Selection algorithm. If the input is  $(k', \ell)$ -nearly sorted for  $k' > k$ , then we can still bound the number of runs that will be produced – it is not hard to see, by partitioning  $R$  into consecutive subsequences so that each of them is  $(k, \ell)$ -nearly sorted, that the number of runs is at most  $k'/k + 1$  (the “+1” refers to the tuples remaining in memory in the end). Similarly, using the fall-back mechanism of Algorithm 2 described below would cause no more than  $k'/k$  overflows (however here more care is needed in handling the overflows).

A more careful analysis of Algorithm 3 would reveal that its probability for a false positive decays exponentially in  $k'/k$ , and so whenever it accepts, the expected risk of running Algorithm 1 or Algorithm 2 (instead of an algorithm more optimized for the completely unsorted case) is still small, because we would most likely still have a small number of runs allowing for an efficient merge. A similar decay in the error probability holds for Algorithm 6. Finally, note that in some instances it may even be beneficial to run into the fall-back mode of our sorting algorithm on purpose, if the resulting number of runs  $r$  would be small enough to make an  $r$ -way merge more efficient than a multi-pass sorting algorithm.

## 7.3 Recovering from a fail mode of the improved sort algorithm

When Algorithm 2 reaches a failure mode in the first pass (we recall that it will not reach a failure mode in the second pass unless one was reached in the first pass) we cannot just dump the contents of  $G$  to a temporary file and continue from where we were.

An appealing direction would be then to try to partition to input file into consecutive segments where on each of them we can run Algorithm 2, and pipe their second phase input to a merge sort algorithm. This will not work with a multi-way merge sort algorithm as per [5], but it will work with the iterated 2-way sort algorithm of [6].

To follow on this, when  $S$  becomes empty and we reach the fail state of the first pass, we write  $G$  to a temporary file and start over from where we were. We also write down the location in  $R$  where the fail state was reached, and reset *last\_handled* to MIN\_VAL. Then, for the second pass, we have a partition of the original relation into subsequences, and a collection of runs from  $G$ , so that each pair of an original subsequence and a corresponding run written from  $G$  can be merged into a sorted run as per the second

stage of Algorithm 2 (without a failure mode being reached). In terms of Section 7.2 we would have  $k'/k + 1$  such segments, which we pipe in turn to the iterated 2-way merge sort algorithm.

The reason that this will not work with a multi-way merge algorithm is the memory requirement of holding  $O(k + \ell)$  tuples for “decoding” each sorted run. The only way to enable multi-way merge is in effect to revert to Algorithm 1 after the first time a fail state was reached. Starting from the second run, we just write it in its entirety to a temporary file. Then, under reasonable memory assumptions we can merge all the runs at once, with the first run requiring us to hold  $O(k + \ell)$  tuples in memory to decode, and each subsequent run requiring enough memory to hold the  $O(1)$  tuples that are buffered from the respective file.

## 7.4 Solid State Drives

Another issue to consider is how our algorithm can perform when the database is not stored on a traditional harddisk drive, but rather on a Solid State Drive (SSD). While current SSDs are mostly restricted to small appliance devices and high-end laptops, one can imagine a time where the technology would advance enough to supplant traditional drives, so it is not too soon to consider database algorithms designed for this new hardware profile.

There are two crucial differences between traditional drives and SSDs. The first is that with an SSD the seek time is negligible, and so a sequential read operation takes the same time as a random access read operation (however when many operations are involved, block sizes may become an issue). In fact seek times are recently becoming less and less relevant also for traditional disk drives under a modern operating system [12]. The second difference is that with some SSD devices writing is very costly relative to reading.

In this context our algorithm should be compared against the algorithm from [5], which performs an  $O(n/\ell)$ -way merge sort, where  $n$  is the size of  $R$  and  $\ell$  is the size of the runs produced by using quicksort on consecutive subsequences. This is a 2-pass algorithm for the practical purposes of current hardware memory size, where the second pass uses random access reads. In the following we assume that the relations are sufficiently nearly-ordered for our algorithms to work.

When comparing sort operations, Algorithm 2 outperforms the multi-way merge sort algorithm. Both our algorithm and the algorithm of [5] perform two passes of reading the full relation. However, while our algorithm writes nothing on disk apart from the final output (which can also be piped for further processing), the multiway merge sort

algorithm has to first rewrite the entire relation as a sequence of sorted runs, which are then read for the merging stage.

The comparative analysis of Algorithm 2 also holds for the evaluation of other operators as in Chapter 5. Both the output of Algorithm 2 and the output of (the second stage of) the multiway merge algorithm can be directly piped to the operation at hand, but our algorithm saves the writing of the entire relation as sorted runs. In fact the only instance in which the algorithm of [5] is significantly better is when (say, due to an unlikely error of the near-sortedness test) we tried to run our algorithm on an input that is  $\Theta(n)$ -far from being  $\ell$ -globally sorted.

There is additionally the question of testing whether our algorithm applies to a given relation, *i.e.*, testing whether the relation is nearly sorted. In SSDs the situation is even better than that of traditional drives, because the testing algorithm in fact fully depends on random access read operations.

## 7.5 Parallel scalability

Our sorting algorithms for nearly sorted inputs can in fact be scaled to a parallel implementation as per the following sketch (we omit the fall-back procedure in case the algorithm fails).

To further understanding, we present the parallel implementation for Algorithm 1. A parallel version for Algorithm 2 can be derived from the following in much the same fashion as the original derivation of Algorithm 2 from Algorithm 1, by deferring all output to the final stage.

Assume that we have  $d$  processors, each with  $O(k + \ell)$  memory (where we need to sort a  $(k, \ell)$ -nearly sorted file), and with access to the whole file. We also assume that  $d$  is small with respect to  $k$  and  $\ell$ , and that all processors are trustworthy. The input file is partitioned into  $d$  consecutive equal size segments, each processor receiving charge of one segment. Each processor performs an independent run of the sorting algorithm for  $(k, \ell)$ -nearly sorted input on its segment. However, also the first  $\ell + k$  and the last  $\ell + k$  records that were supposed to be output (to the respective intermediate file) are not output, but instead are collected along with the up to  $k$  out of order records. Exceptions are the beginning of the very first segment and the end of the very last segment, that are still written to their segment's output. A copy of the first and last record that was output for each segment is also kept for further checks, and then all segments are considered to be concatenated (this means that future sequential reads may require up to  $d - 1$  additional seeks).

If none of the processors failed, we check whether the concatenation of the output

segments is sorted (for this we kept the first and last record in each segment's output). If this check has passed then we may continue. We sort all the records that were not output yet. This requires  $\tilde{O}(d(k + \ell))$  computation and communication to pass records between processors so that each processor gets a consecutive segment from the sorted in-memory records, by proceeding along the following steps:

1. A designated processor chooses uniformly at random  $t = O(d \log d)$  numbers in  $\{1, \dots, m\}$ , where  $m$  is the total number of in memory records.
2. The processors holding the corresponding records report them. Let  $r_1, \dots, r_t$  denote the values of their key attributes in sorted order, let  $r_0 = \text{MIN\_VAL}$  and let  $r_{t+1} = \text{MAX\_VAL}$ .
3. Each processor reports how many records it holds with key values between  $r_i$  and  $r_{i+1}$  for  $0 \leq i \leq t$ .
4. A designated processor calculates  $0 = i_0 < i_1 < \dots < i_d = t + 1$  such that there are  $O(m/d)$  records between  $r_{i_j}$  and  $r_{i_{j+1}}$  for all  $0 \leq j < d$ ; with high probability such  $i_j$  exist, and otherwise Step 1 above is restarted.
5. For every  $j = 1, \dots, d$  all processors communicate their records between  $r_{i_{j-1}}$  and  $r_{i_j}$ , which processor  $j$  stores in order as its assigned records.

Finally, the records in memory are merged with the previous output. This can also be parallelized, after a preliminary binary search is performed over the previous intermediate output to assign to each processor a segment into which its records will be merged (one would expect the assigned segment boundaries to typically resemble the boundaries of the original output segments, but this cannot be analytically guaranteed). Note that although this procedure is only guaranteed for  $(k, \ell)$ -near sorted relations, it may also work for sufficiently “evenly spread”  $(O(kd), \ell)$ -nearly sorted relations.

## 7.6 Preliminary experiments about natural occurrence of near-sortedness

We built and ran some TPC-C benchmarks according to the guidelines in the TPC-C standard [7] using a database generated according to Chapter 3 of the “Installation and User Guide” [7]. During the benchmark runs, we monitored the sort operations, and for each execution of a sort operation we checked the nearly-sortedness condition of the relation being sorted for various parameters  $k$  and  $\ell$ . Our tests showed that for

$k = \ell = \lceil \sqrt{n} \rceil$ , where  $n$  is the number of tuples, more than 90% of the relations were  $(k, \ell)$ -nearly sorted before the sorting started. These values for  $k$  and  $\ell$  were selected because they fit the case where for a fixed  $n$  they would minimize the required working memory for the testing algorithm paired with the sorting algorithm; in fact for the current systems the amount of working memory would accommodate higher values of  $k$ . These are very preliminary experimental results. We intend to build a fuller implementation of our algorithms inside a working database and report on it in a future work.

## Bibliography

- [1] N. Ailon, B. Chazelle, S. Comandur, and D. Liu. Estimating the distance to a monotone function. *Random Struct. Algorithms*, 31(3):371–383, 2007.
- [2] F. Ergün, S. Kannan, R. Kumar, R. Rubinfeld, and M. Viswanathan. Spot-checkers. *J. Comput. Syst. Sci.*, 60(3):717–751, 2000.
- [3] E. Fischer. The art of uninformed decisions: A primer to property testing. In G. Paun, G. Rozenberg, and A. Salomaa, editors, *Current Trends in Theoretical Computer Science: The Challenge of the New Century*, volume I, pages 229–264. World Scientific Publishing, 2004.
- [4] E. Fischer, E. Lehman, I. Newman, S. Raskhodnikova, R. Rubinfeld, and A. Samorodnitsky. Monotonicity testing over general poset domains. In *STOC*, pages 474–483, 2002.
- [5] H. Garcia-Molina, J. D. Ullamn, and J. Widom. *Database Systems*, section 15.4.1, Two-Phase, Multiway Merge-Sort, pages 723 – 725. Prentice Hall, New Jersey, 2009.
- [6] D. E. Knuth. *The art of Computer Programming*, section 5.4.1, Multiway Merging and Replacement Selection, pages 252 – 263. Addison Wesley, third edition, 1998.
- [7] D. R. Llanos. Tpc-c-uva: An open-source tpc-c implementation for global performance measurement of computer systems. *ACM SIGMOD Record*, December 2006. ISSN 0163-5808.
- [8] M. Parnas, D. Ron, and R. Rubinfeld. Tolerant property testing and distance approximation. *J. Comput. Syst. Sci.*, 72(6):1012–1042, 2006.
- [9] D. Ron. Property testing (a tutorial). In S. Rajasekaran, P. M. Pardalos, J. H. Reif, and J. D. P. Rolim, editors, *Handbook of Randomized Computing*. Kluwer Press, 2001.
- [10] R. Rubinfeld. Sublinear time algorithms. In *International Congress of Mathematicians*, volume III, pages 1095–1110. European Mathematical Society, 2006.



- [11] E. K. S. Halevy. Distribution-free property testing. In *Proc. RANDOM*, pages 302–317, 2003.
- [12] C. Staelin. Disk I/O in Linux. Technical Report HPL-2002-352, Hewlett-Packard Laboratories, 2002.

# Appendix A

## A.1 A simple sortedness test

We present here a simple  $([0, k], [\ell, \ell])$ -sortedness test that makes a sub-linear number of read operations. In the standard terminology of property testing, this is equivalent to a  $k/n$ -tester for the property of being  $\ell$ -globally sorted, that makes a sub-linear number of queries. While the analysis of Algorithm 3 above is self-contained, this and the following section can be read as a warm-up for the more complex analysis presented there.

Our testing algorithm, Algorithm 7, outputs ACCEPT (accepts) with probability 1 if the relation  $R$  is  $\ell$ -globally sorted, and outputs REJECT (rejects) with probability at least  $2/3$  if  $R$  is  $k$ -far from being  $\ell$ -globally sorted. Since the algorithm always accepts  $\ell$ -globally sorted inputs, the probability of detecting  $k$ -far ones can be amplified easily by running the algorithm several times and rejecting if any of the runs rejected.

Algorithm 7 is a variant of the algorithms from [11, 2] for testing monotonicity. In each iteration of the algorithm a *center point* is selected uniformly at random, and the algorithm looks for a violation of the monotonicity with this center point. The search for a violation is done by randomly sampling from growing neighborhoods of the center point, but without checking neighbors within distance at most  $\ell$ .

**Theorem A.1.** *Algorithm 7 is a  $([0, k], [\ell, \ell])$ -sortedness test, that makes  $O\left(\frac{n}{k}(\log n + \ell)\right)$  queries and errs with probability at most  $1/3$ .*

To prove this theorem, we need several lemmas and definitions. The following is an immediate conclusion of Definition 2.5

**Lemma A.2.** *Assume that  $R$  is  $k$ -far from being  $\ell$ -globally sorted. Let  $S \subset [n]$  be a set such that for every violation  $(i, j)$  of the  $\ell$ -global sortedness of  $R$  either  $i \in S$  or  $j \in S$ . Then  $|S| > k$ .*

---

**Algorithm 7** ( $([0, k], [\ell, \ell])$ -sortedness test)

---

```

for  $2n/k$  times do
  pick  $i \in [n]$  uniformly at random
  for  $a = \ell + 1$  to  $3\ell$  do
    if  $R[i] < R[i - a]$  or  $R[i] > R[i + a]$  then
      return REJECT
    end if
  end for
  for  $h = \lfloor \log \ell + 1 \rfloor$  to  $\lceil \log i \rceil$  do
    for 21 times do
      pick  $a \in [\ell + 1, 2^h]$  uniformly at random
      if  $R[i] < R[i - a]$  or  $R[i] > R[i + a]$  then
        return REJECT
      end if
    end for
  end for
end for
return ACCEPT

```

---

**Definition A.3.** For every pair  $(i, j)$  that violates  $\ell$ -global sortedness in  $R$  we define two subsets of indices  $A_{ij}, A_{ji} \subseteq [i, j]$  as follows:

$$A_{ij} \triangleq \{j' \in [i + \ell, j] : R[i] > R[j']\}$$

$$A_{ji} \triangleq \{i' \in [i, j - \ell] : R[i'] > R[j]\}$$

We say that  $i$  is active in the violating pair  $(i, j)$  if  $|A_{ij}| > |A_{ji}|$ , and otherwise we say that  $j$  is active in it.

Notice that by definition, in any violating pair  $(i, j)$  one of  $i$  or  $j$  must be active. We also have the following lemma.

**Lemma A.4.** For any violating pair  $(i, j)$ , we have  $|A_{ij}| + |A_{ji}| \geq j - i - 2\ell$ . Consequently, if  $i$  (respectively  $j$ ) is active in this pair, then the size of  $A_{ij}$  (respectively  $A_{ji}$ ) is at least  $\frac{j-i}{2} - \ell$ .

*Proof.* Note that the claim trivially holds unless  $j - i > 2\ell$ . Otherwise, it is easy to see that every  $k \in [i + \ell, j - \ell]$  belongs to  $A_{ij}$  or  $A_{ji}$  (or both). Hence the claim follows.  $\square$

We define the *active set* of  $R$ , denoted by  $A(R)$ , as the set of all entries in  $R$  that are active in some violation. From Lemma A.2 and the observation that in any violating pair one of the indices is active, we obtain the following lemma.

**Lemma A.5.** If  $R$  is  $k$ -far from being  $\ell$ -globally sorted then  $|A(R)| > k$ .

*Proof of Theorem A.1.* The probe complexity of Algorithm 7 and the fact that every  $R$  that is  $\ell$ -globally sorted is always accepted are clear from its description. To complete the proof, we assume that  $R$  is  $k$ -far from being  $\ell$ -globally sorted and show that Algorithm 7 rejects it with probability at least  $2/3$ .

The algorithm may fail to detect that  $R$  is not  $\ell$ -globally sorted if either one of the following two events occurs:

1. None of the  $2n/k$  points sampled by the algorithm is in the active set  $A(R)$
2. The algorithm picked at least one point  $i \in A(R)$ , but failed to detect that  $i$  belongs to some violation.

It is easily verified that the probability of the first event is at most  $(1 - k/n)^{2n/k} \leq e^{-2} < 1/6$ . We now bound the probability of the second event. By the definition of  $A(R)$ , for every  $i \in A(R)$  there is an index  $j$  such that either  $(i, j)$  or  $(j, i)$  is a violation of  $\ell$ -global sortedness, and  $i$  is active in this violation. Assume without loss of generality that  $(i, j)$  is a violation. If  $|j - i| \leq 3\ell$  then this violation is discovered with probability 1. Otherwise, since  $i$  is active in this pair, by Lemma A.4 we have  $|A_{ij}| \geq \frac{j-i}{2} - \ell \geq \frac{j-i}{6}$ . For  $h = \min\{b : 2^b > j - i\}$ , we claim that at least a  $1/12$ -fraction of the pairs  $\{(i, c) : i < c \leq i + 2^h\}$  are violating. This is due to the fact that  $j - i > 2^{h-1}$  and  $|A_{ij}| \geq \frac{j-i}{6}$  (i.e., for at least a  $1/6$ -fraction of the points  $c$  between  $i$  and  $j$  the pair  $(i, c)$  is a violation). The probability that the algorithm fails to find a violation for this  $h$  is at most  $(11/12)^{2^h} \leq 1/6$ , and hence the probability of the second event is at most  $1/6$ , implying that the total probability that the algorithm will accept an  $R$  that is  $k$ -far from being  $\ell$ -globally sorted is at most  $1/3$ .  $\square$

## A.2 A sortedness test with better probe complexity

We can improve the probe complexity of the previous tester to  $O(\frac{n}{k} \log n)$ , which is independent of  $\ell$ , if we consider the relaxed task of distinguishing between  $\ell$ -globally sorted relations and those that are  $k$ -far from being  $3\ell$ -globally sorted. Namely, given random access to a relation  $R$ , the new tester should output ACCEPT if  $R$  is  $\ell$ -globally sorted, and it should output REJECT with probability at least  $2/3$  if  $R$  is  $k$ -far from being  $3\ell$ -globally sorted.

To achieve the improvement in probe complexity we need to slightly modify Algorithm 7 by removing the procedure that checks for violations between “close” pairs of indices. The new algorithm is formally described as follows:

**Theorem A.6.** *Algorithm 8 is a  $([0, k], [\ell, 3\ell])$ -sortedness test that makes  $O(\frac{n}{k} \log n)$  queries and errs with probability at most  $1/3$ .*

---

**Algorithm 8** (tester for  $\ell$ -global sortedness with improved probe complexity)
 

---

```

for  $2n/k$  times do
  pick  $i \in [n]$  uniformly at random
  for  $h = \lceil \log \ell + 1 \rceil$  to  $\lceil \log i \rceil$  do
    for 21 times do
      pick  $a \in [\ell + 1, 2^h]$  uniformly at random
      if  $R[i] < R[i - a]$  or  $R[i] > R[i + a]$  then
        return REJECT
      end if
    end for
  end for
end for
return ACCEPT
  
```

---

Similarly to the proof of Theorem A.1, the only non-trivial part that has to be proved here is that if  $R$  is  $k$ -far from being  $3\ell$ -globally sorted then Algorithm 8 rejects it with probability at least  $2/3$ . But first we need the following lemma, which is almost identical to Lemma A.4 above.

**Lemma A.7.** *Let  $i, j \in [n]$  be such that the pair  $(i, j)$  violates the  $3\ell$ -global sortedness of  $R$ . Then there exist sets  $A_{ij}, A_{ji} \subseteq [i, j]$  such that*

1. *For all  $a_i \in A_{ij}$ , the pair  $(i, a_i)$  violates the  $\ell$ -global sortedness of  $R$ .*
2. *For all  $a_j \in A_{ji}$ , the pair  $(a_j, j)$  violates the  $\ell$ -global sortedness of  $R$ .*
3. *At least one of  $A_{ij}$  or  $A_{ji}$  contains at least  $\frac{j-i-2\ell}{2}$  elements.*

*Proof.* Let  $(i, j)$ ,  $i \leq j - 3\ell$  be a violating pair, and define  $A_{ij}$  and  $A_{ji}$  as in Definition A.3. It is easy to see that every  $s \in [i + \ell, j - \ell]$  belongs to  $A_{ij}$  or  $A_{ji}$  (or both) as above. Hence the claim follows since there are  $j - i - 2\ell$  possible values for  $s$   $\square$

Notice that the sets  $A_{ij}$  and  $A_{ji}$  are defined with respect to a pair  $(i, j)$  that violates  $3\ell$ -global sortedness, but they contain indices that (together with  $i$  or  $j$ ) violate only  $\ell$ -global sortedness. As before, we will say that  $i$  is active in the pair if  $|A_{ij}| \geq |A_{ji}|$  and otherwise we say that  $j$  is active. Observe that for any pair  $(i, j)$  that violates  $3\ell$ -global sortedness, one of  $i$  or  $j$  is active and so the corresponding set  $A_{ij}$  or  $A_{ji}$  is of size at least  $\frac{1}{6}(j - i)$ . From here, we can complete the proof of Theorem A.6 similarly to the proof of Theorem A.1.

*Proof of Theorem A.6.* Assume that  $R$  is  $k$ -far from being  $3\ell$ -globally sorted, and consider the set  $A$  of all active indices in the pairs that violate the  $3\ell$ -global sortedness of  $R$ . According to Lemma A.2,  $|A| \geq k$ . The algorithm may still accept  $R$  if either one of the following two events occurs:

1. None of the  $2n/k$  points sampled by the algorithm is in the active set  $A$ ;
2. The algorithm picked at least one point  $i \in A$ , but failed to detect that  $i$  belongs to some pair that violates the  $\ell$ -global sortedness of  $R$ .

Similarly to the proof of Theorem A.1, the probability of the first event is at most  $1/6$ . Now we bound the probability of the second event. By the definition of  $A$ , for every  $i \in A$  there is an index  $j$  and a set  $A_{ij}$  of indices such that

- For all  $a_i \in A_{ij}$ , the pair  $(i, a_i)$  violates the  $\ell$ -global sortedness.
- All indices  $a_i \in A_{ij}$  reside between  $i$  and  $j$ .
- The size of  $A_{ij}$  is at least  $\frac{|j-i|}{6}$ .

For  $h = \min\{b : 2^b > |j - i|\}$ , we claim that at least a  $1/12$ -fraction of the pairs  $\{(i, c) : 0 < |c - i| \leq 2^h\}$  violate  $\ell$ -global sortedness. This is again due to the fact that  $|j - i| > 2^{h-1}$ , and because  $|A_{ij}| \geq \frac{|j-i|}{6}$ . The probability that the algorithm fails to find a violation for this  $h$  is at most  $1/6$ , and hence the total probability that the algorithm will accept an  $R$  that is  $k$ -far from being  $3\ell$ -globally sorted is at most  $1/3$ .  $\square$

**Remark A.8.** *It is not hard to see that for any fixed  $\mu > 0$ , we can also get a  $([0, k], [\ell, (2+\mu)\ell])$ -sortedness test that makes  $O(\frac{n}{k} \log n)$  queries.*

# שימוש בבדיקת תכונות לזיהוי יעיל של רלציות אשר קרובות להיות ממוינות

תזה

הוגשה לשם מילוי חלקי של הדרישות לקבלת התואר  
למגיסטר למדעים

שגיא בן משה

הוגשה לסנט הטכניון – מכון טכנולוגי לישראל

מרץ 2010

חיפה

אדר התשי"ע

המחקר נעשה בהנחיית פרופסור חבר אלדר פישר  
בפקולטה למדעי המחשב

אני מודה למנחה שלי – אלדר פישר, שהנחה אותי במשך  
השנתיים האחרונות, על התמיכה והעידוד שלו ועל  
הרעיונות וההצעות החשובות שלו. אני רוצה גם להודות  
לשותפי לעבודה: ירון קנזה, אריה מצליח, קרל סטלין,  
מאני פישר.

אני מודה לעזרה הפיננסית הנדיבה של הטכניון ושל קרן  
האיחוד האירופי ERC-2007-StG



## הקדמה

מרבית הפעולות הנפוצות על רלציות (קבצים) של מסד נתונים ניתנות לביצוע יעיל במיוחד כאשר הרלציות ממוינות לפי מפתח מיון רלוונטי. לדוגמה, צירוף טבעי (Natural Join) בשאלתה במסדי נתונים ניתן לביצוע על ידי מעבר סידרתי אחד על הרלציות כאשר הרלציות ממוינות לפי התכונות המשותפות בצירוף. עם זאת, רוב הרלציות לא יכולות להיות שמורות בצורה ממוינת בגלל העלות היקרה של תיחזוק הרלציות בצורה זאת, או הימצאותם של כמה מפתחות מיון אפשריים ברלציה.

על כן, שאילתות במבני נתונים צריכות לרב לטפל ברלציות אשר שמורות בצורה לא ממוינת על הדיסק. כאשר המעבד צריך גישה לרשומות של הרלציה בסדר מסויים הוא יכול לבצע זאת על ידי גישה דרך האינדקס, אם האינדקס קיים, או על ידי מיון קודם של הרלציה. אם המערכת היתה אמורה לשמור את הרלציה בצורה ממוינת באופן קבוע אז הכנסות ועדכונים ברלציה היו מתבצעים בצורה מאוד לא יעילה, ובנוסף לכך שמירת הרלציה באופן ממוין אינה אפשרית כלל כאשר יש מספר מפתחות מיון שונים.

מסתבר שהרבה פעמים רלציות קרובות להיות ממוינות במציאות, במובן זה שמרבית הרשומות יהיו במקום קרוב למקומן ה"טבעי" לפי סדר המיון הרצוי. רלציות כאלו יכולות לדוגמה להיוצר על ידי התרחישים הבאים:

1. רלציה שנשמרה בצורה ממוינת ועדכנו בה או הוסיפו לה מספר רשומות.
2. רלציה שנוצרה מפעולה קודמת כגון צירוף טבעי של רלציות.
3. רלציה שהיתה ממוינת, על מפתח מיון אחד, ובגלל התאמה טבעית בין התכונות שלה כגון שכר דירה וגודל, או גובה ומשקל, קרובה להיות ממוינת על מפתח מיון אחר.

נכון להיום ההתייחסות היא זהה עבור הרלציות הקרובות להיות ממוינות והרלציות הרחוקות מכך, ובכל פעולה על רלציה משתמשים באלגוריתם כללי. ההפסד העיקרי מכך הוא שפעולות על הרלציות הקרובות להיות ממוינות היו יכולות להתבצע בצורה הרבה יותר יעילה על ידי אלגוריתם שמנצל את התכונה הזאת. במחקר זה אנו מראים בצורה מפורשת אלגוריתם שממין רלציה הקרובה להיות ממוינת בשני מעברים סידרתיים של קריאה של הרלציה, כאשר ביצוע פעולה זאת באמצעות אלגוריתם כללי דורש משאבים גדולים בהרבה.

כדי להרויח מתכונה זאת המערכת צריכה גם לזהות מתי להפעיל את האלגוריתמים שיודעים לנצל אותה ומתי להפעיל את האלגוריתמים הכלליים. לכן יש צורך בבדיקה הסתברותית יעילה שבדקת האם התכונה המבוקשת קיימת ברלציה.

אנחנו מגדירים "קרוב להיות ממוין" באמצעות שני פרמטרים  $k$  ו  $l$ .

הפרמטר  $l$  מגדיר לנו עד כמה מדויק צריך להיות המיון,  $l$  גדול יותר אומר שהרשומות יכולות להיות רחוקות יותר מהמקום הסופי שלהם ברלציה הממוינת.

הפרמטר  $k$  מגדיר לנו כמה רשומות יכולות להיות מחוץ למקום שלהם באופן מוחלט, זאת אומרת מכמה רשומות צריך להתעלם לפני שהרלציה מצייתת לפרמטר  $l$ .

על מנת לבצע את הבדיקה אנחנו משתמשים ברעיונות מתחום בדיקת התכונות (Property Testing). בדיקת תכונות היא תחום מחקרי העוסק באלגוריתמים שמבססים את הפלט שלהם על קריאת חלק קטן בלבד מהקלט (בגישה אקראית). אלגוריתמים אלו מבדילים, עם סיכוי קטן לשגיאה, בין קלטים המקיימים תכונה רצויה לבין אלו הרחוקים מלקיים את התכונה.

במקרה שלנו אנחנו דורשים גם שזמן החישוב יהיה נמוך ככל הניתן. אנחנו מתמודדים עם שני פרמטרים  $k$  ו  $l$  במקום עם פרמטר אחד. בנוסף לכך אנחנו דורשים שהבדיקה תהיה סובלנית, וזאת מכיוון שאנחנו רוצים להבטיח קבלה של קלטים שבהם  $k$  ו  $l$  גדולים מאפס אך עדיין קטנים מספיק, ולא רק דחייה של קלטים בהם  $k$  ו  $l$  גדולים מדי. דרישה זאת מבטיחה לנו שהבדיקה תקבל יותר קלטים שקרובים להיות ממוינים וניתן עבורם להשתמש באלגוריתמים שמנצלים את התכונה הזאת. לבסוף אלגוריתם הבדיקה שלנו יכול גם להצהיר מראש על כל הקריאות המתבצעות, דבר שמייעל את זמן הגישה הכולל.

### הגדרות פורמליות

רלציה מוגדרת להיות  $k$  קרובה להיות ממוינת כאשר ניתן להוריד  $k$  רשומות מהרלציה ולקבל רלציה ממוינת.

רלציה מוגדרת להיות  $l$  ממוינת גלובלית כאשר כל שתי רשומות במרחק קטן מ  $l$  מקיימות את סדר המיון.

רלציה מוגדרת להיות  $k$  קרובה להיות  $l$  ממוינת גלובלית, או בקיצור  $(k,l)$ -כמעט ממוינת, כאשר קיימות לא יותר מ  $k$  רשומות שמחוץ להן כל הרשומות מקיימות את התנאי עבור רלציה  $l$  ממוינת גלובלית.

דוגמאות להגדרות :

1	2	3	4	5	6	7	8	9	10
---	---	---	---	---	---	---	---	---	----

$R_1$  ממוין

1	8	3	4	5	6	7	2	9	10
---	---	---	---	---	---	---	---	---	----

$R_2$  2 קרוב להיות ממוין

1	4	3	2	5	6	8	7	9	10
---	---	---	---	---	---	---	---	---	----

$R_3$  3 ממוין גלובלית

9	4	3	2	5	6	8	7	1	10
---	---	---	---	---	---	---	---	---	----

$R_4$   $(2,3)$ -כמעט ממוין

כמו שניתן לראות, הרלציה הראשונה ממוינת. הרלציה השנייה 2 קרובה להיות ממוינת מכיוון שאם מורידים את 8 ו 2 מהרלציה היא נהיית ממוינת. הרלציה השלישית היא 3 ממוינת גלובלית מכיוון שכל שתי רשומות ברלציה שהמרחק ביניהן הוא לפחות 3 מקיימות את הסדר הנכון. הרלציה הרביעית היא  $(2,3)$ -כמעט ממוינת מכיוון שלמעט הרשומות 9 ואת 1 היא עומדת בתנאים להיות 3 ממוינת גלובלית.

$([k_1, k_2], [l_1, l_2])$ -בודק מיון עם סיבוכיות קריאה  $q$  והסתברות לשגיאה  $\delta$  מוגדר כאלגוריתם המבצע מקסימום  $q$  קריאות מהקלט ומקיים את התנאים הבאים :

1. אם הרלציה היא  $(k_1, l_1)$ -כמעט ממוינת אז האלגוריתם מקבל (עונה "כן") בהסתברות של לפחות  $1-\delta$ .
2. אם הרלציה היא אפילו לא  $(k_2, l_2)$ -כמעט ממוינת אז האלגוריתם דוחה (עונה "לא") בהסתברות של לפחות  $1-\delta$ .

### האסטרטגיה

כמו שנכתב בהקדמה בדי"כ ניתן למצוא  $k$  ו  $l$  (אשר יהיו שמישים ביחס לזיכרון המחשב שברשותינו) עבורם רלציה טיפוסית תהיה בהרבה מקרים  $(k, l)$ -כמעט ממוינת. האסטרטגיה לביצוע שאילתות תשתמש בכלים הבאים אותם פיתחנו במהלך המחקר :

1. אלגוריתם לביצוע מיון בצורה יעילה עבור רלציות  $(k, l)$ -כמעט ממוינות. השימוש בפעולות המיון הינו דוגמה אחת ואנחנו משתמשים בדוגמה זאת מכיוון שהיא הכי קלה להסבר מבין פעולות מסד הנתונים. חשוב לזכור שאלגוריתם המיון חייב לקחת בחשבון כישלון, מכיוון שהבודק אותו פיתחנו הוא הסתברותי ולכן יכול להיווצר מצב בו אלגוריתם המיון קיבל קלט אשר אינו עומד בתנאים.
  2. בודק יעיל וסובלני לבדיקת מצב של כמעט ממוינות. הבודק מקבל כאשר הרלציה היא  $(k, l)$ -כמעט ממוינת והוא דוחה כאשר הרלציה אפילו לא  $(6k, 6l)$ -כמעט ממוינת.
- עם מרכיבים אלו נוכל לבצע שאילתות לפי הגישה הבאה :

1. נפעיל את הבודק על הרלציה עם פרמטרים  $k$  ו  $l$ .
2. אם הבודק דוחה נעבור למיון באמצעות האלגוריתם הכללי הרגיל.
3. אם הבודק מקבל :
  - א. נפעיל את אלגוריתם המיון המיוחד עם הפרמטרים  $6k$  ו  $6l$ .
  - ב. אם אלגוריתם המיון המיוחד הצליח, סיימנו.
  - ג. אם אלגוריתם המיון נכשל, ננסה לנצל את הפלט שכבר קיבלנו תוך כדי חזרה לאלגוריתם הכללי הרגיל.

כאשר הבודק מופעל, הסיכוי להגיע לשלב 3 (ג) נמוך ביותר. בנוסף לכך, עלות הבודק על רלציות שאינן  $(k, l)$ -כמעט ממוינות היא קטנה, ולפי בדיקות מקדימות שערכנו בהרבה מקרים הרלציות הן  $(k, l)$ -כמעט ממוינות עבור ערכים רלוונטים של  $k$  ו  $l$ . לעומת זאת, הרווח על ביצוע מיון ברלציה שמקיימת את התנאים גבוה ביותר, ולכן הרווח הכללי בשימוש באסטרטגיה זאת מצדיק את העלות הנוספות של הבדיקה.

### אלגוריתם המיון

אנחנו מציגים שני אלגוריתמי מיון דומים זה לזה.

האלגוריתם הראשון ממיין רלציה  $(k, l)$ -כמעט ממוינת על ידי שני מעברים סידרתיים על הרלציה. במעבר הראשון האלגוריתם מתנהג בדומה לאלגוריתם הידוע Replacement Selection. במעבר זה האלגוריתם קורא כמה שיותר רשומות לזיכרון ובכל שלב פולט את הרשומה הנמוכה ביותר שעדיין גבוהה מהרשומות שעוד לא נפלטו (לקובץ זמני), תוך כדי קריאת הרשומה הבאה מקובץ הקלט. במעבר השני הרשומות שנשארו בזיכרון מוכנסות למיקום הסופי שלהם תוך כדי מעבר על הפלט של השלב הראשון.

במקרה שהאלגוריתם נכשל (כאשר הקלט לא היה  $(k, l)$ -כמעט ממוין) אנו מקבלים מיון של חלק מהרשומות, ואפשר עתה להתחיל את האלגוריתם מהמקום בו נעצרנו, בדומה לאלגוריתם Replacement Selection רגיל, ולבסוף למיין על ידי מיזוג.

האלגוריתם השני עובד כמו האלגוריתם הראשון, אבל בשינוי מרכזי אחד. בשלב הראשון האלגוריתם אוסף את הרשומות שהאלגוריתם הראשון היה משאיר בזיכרון, אולם הוא לא כותב כלל לדיסק. בשלב השני האלגוריתם למעשה חוזר על השלב הראשון, אולם עתה הוא פולט את כל הרשומות בצורה ממוינת בשילוב אלו שנשמרו בזיכרון. בצורה זאת, האלגוריתם חוסך את כתיבת הביניים לדיסק ומסתפק בשתי קריאות סידרתיות של הקלט בלבד, וגם מאפשר בכך הפנייה מיידית של הפלט לעיבוד נוסף במידה וזה דרוש. החיסרון בשיטה זאת הוא הפסד של אופציה ההתאוששות הקלה.

## הבודק

הבודק הינו סובלני, זאת אומרת שהוא מקבל גם קלטים כמעט ממוינים שאינם ממוינים לחלוטין, עבורם אלגוריתם המיון עדיין עובד. אלגוריתם הבדיקה מבדיל בין רלציה  $(k, l)$ -כמעט ממוינת לבין רלציה שאינה  $(6k, 6l)$ -כמעט ממוינת. אנחנו עדיין לא יודעים עד כמה ניתן לרדת מהמקדם 6, אבל ברור לנו שלא ניתן להבדיל ב  $k$  בצורה מדוייקת. לעומת זאת אנחנו יודעים שניתן להבדיל ב  $l$  בצורה מדוייקת, אבל אז סיבוכיות הבודק שבידינו תהיה תלויה ב  $O(l)$  ואנו נעדיף לחסוך בעלות זאת.

הבודק אותו פיתחנו הוא לא אדפטיבי, מה שאומר שניתן להחליט אילו רשומות לקרוא לפני שקוראים אפילו רשומה אחת. ההחלטה היחידה שתלויה בערכים שנקראו היא ההחלטה הסופית האם לקבל או לדחות. דבר זה מעלה את הביצועים מכיוון שניתן לשלוח את כל בקשות הקריאה למערכת ההפעלה מראש, וזו יכולה לסדר אותן מחדש באופן אופטימלי.

סיבוכיות הבודק היא  $O((n/k)\log((n/k))\log(n)\log(\log(n)))$ , ואינה תלויה ב  $l$  לשם המחשה עבור  $k=O(n)$  ביטוי התלות יתן לנו  $\tilde{O}(\log(n))$

## אספקטים נוספים ותוכניות עתידיות

כאשר מסתכלים על התקני חומרה חדשים כגון דיסק מצב מוצק (Solid State Drive), עלות הקריאה האקראית יורדת בצורה משמעותית, וההבדל העיקרי בביצועים הוא בין מהירות הקריאה למהירות הכתיבה. בשיטות אלו אלגוריתם המיון השני אותו פיתחנו משפר את הביצועים בצורה משמעותית, מכיוון שבניגוד לכל השיטות הידועות כיום האלגוריתם אותו פיתחנו לא רושם מידע זמני לדיסק והוא ממיין את הקלט על ידי שתי קריאות סידרתיות בלבד. חשוב לציין שבגלל העלות הזניחה של קריאה אקראית מהקובץ בהתקנים מסוג זה, עלות הבודק אותו פיתחנו זולה אף יותר.

אלגוריתמי הבדיקה והמיון אותם פיתחנו ניתנים להתאמה למודל מבוזר, שבו כל מעבד אחראי על חלק (רציף) אחר בקלט. בשלב המיון הראשון כל מעבד יכול לעבוד על החלק שלו, ואת הרשומות הנמצאות מחוץ למקומן המעבדים יעבירו ביניהם כהכנה לשלב השני.

ברצוננו לממש את השיטה ולבצע ניסויים נוספים במערכות מסדי נתונים, במטרה לזהות במדוייק את אחוז הרלציות הכמעט ממוינות עבור  $k$  ו  $l$  רלוונטיים, ולמצוא את אחוז השיפור בפועל המתקבל על ידי השימוש בבודק ובאלגוריתמי המיון המיוחדים. במישור התיאורטי, ברצוננו לגלות עד כמה נמוך ניתן להוריד את מקדם הבודק מתחת ל 6, למצוא דרכים לשיפור זמן המיון כאשר אלגוריתם המיון המיוחד נכשל (במיוחד האלגוריתם השני), ולמצוא תכונות נוספות שניתן לבדוק ולנצל בכדי לשפר פעולות נפוצות במסדי נתונים.