ON SPECTRA OF SENTENCES OF MONADIC SECOND ORDER LOGIC WITH COUNTING

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Abstract. We show that the spectrum of a sentence ϕ in Counting Monadic Second Order Logic (CMSOL) using one binary relation symbol and finitely many unary relation symbols, is ultimately periodic, provided all the models of ϕ are of clique width at most k, for some fixed k. We prove a similar statement for arbitrary finite relational vocabularies τ and a variant of clique width for τ-structures. This includes the cases where the models of ϕ are of tree width at most k. For the case of bounded tree-width, the ultimate periodicity is even proved for Guarded Second Order Logic $GSOL$. We also generalize this result to many-sorted spectra, which can be viewed as an analogue of Parikh's Theorem on context-free languages, and its analogues for context-free graph grammars due to Habel and Courcelle.

Our work was inspired by Gurevich and Shelah (2003), who showed ultimate periodicity of the spectrum for sentences of Monadic Second Order Logic where only finitely many unary predicates and one unary function are allowed. This restriction implies that the models are all of tree width at most 2, and hence it follows from our result.

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Contents

1. Introduction and statement of results

Let τ be a vocabulary, i.e., set of relation and function symbols, and ϕ be a sentence in some fragment of second order logic $SOL(\tau)$. The spectrum $spec(\phi)$ of ϕ is the set of finite cardinalities (viewed as a subset of $\mathbb{N}$), in which ϕ has a model. In 1952 Scholz [Sch52] asked what are the spectra of sentences of first order logic FOL. In 1955 Asser [Ass55] asked whether the complement $\mathbb{N} - spec(\phi)$ is also a spectrum of some FOL-sentence. We note that for SOL-sentences Asser's problem has a trivial solution[1]. For FOL and $MSOL$, both problems are still open[2]. The second problem has been positively answered for certain restricted vocabularies, cf. [DFL97].

In the seventies a series of papers related the first order spectra to complexity theory, cf. [JS72, Fag74a, Fag75, Chr76, LG77, Lyn82].

In the nineties there was renewed interest in first order spectra. Initiated by É. Grandjean's work, [BS87, Gra90], the focus was now on restricted vocabularies. It is known from [Fag74a] that there is a first order sentence involving only one binary relation symbol the spectrum of which is **NEXPTIME**-complete, hence $\mathbf{NP}_1$-complete when the natural numbers are written in unary. It follows from [DR96] that this is also true if the vocabulary consists of two unary function symbols.

It is an easy observation, however, that when the vocabulary contains only unary relation symbols, the spectrum of an FOL-sentence is ultimately constant.

Definition 1.1. *A set $X \subseteq \mathbb{N}$ is* ultimately periodic *if there are $a, p \in \mathbb{N}$ such that for each $n \geq a$ we have that $n \in X$ iff $n + p \in X$.*

In [DFL97] the case with finitely many unary relation symbols and one unary function is studied, and it is shown that those first order spectra are ultimately periodic. In [GS03] this is generalized to

Theorem 1.2 (Gurevich and Shelah)**.** *Let ϕ be a sentence of $MSOL(\tau)$ where τ consists of finitely many unary relation symbols and one unary function. Then $spec(\phi)$ is ultimately periodic,*

Remark 1.3. *Given any ultimately periodic function $f : \mathbb{N} \to 2$ it is easy to construct a regular language L such that the lengths $\mid w \mid$ of the words $w \in L$ are exactly those $n \in \mathbb{N}$ with $f(n) = 1$. Hence, by Büchi's Theorem, cf.* [Str94]*, for every ultimately periodic f there is an existential $MSOL$-sentence ϕ with $spec(\phi) = \{n : f(n) = 1\}$. If we add the existentially quantified set variables as unary predicates to the vocabulary, we also get a FOL-sentence.*

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[1] If $\tau = \{R_1, \ldots, R_n\}$ and $\phi \in SOL(\tau)$ put $\psi = \exists R_1, \ldots, R_n \phi$. Then

$$spec(\phi) = \{n : \{0, 1, \ldots n - 1\} \models \psi\}$$

hence the complement is defined by $\neg\psi$.

[2] The Chinese logician S. K. Mo, [Mo91] announced some progress, but we could not get hold of the paper.

There seem to be a deeper phenomenon hidden here which we know from infinite model theory, cf. [She90]. There one studies the generalized spectrum, i.e. number $spec_T(\kappa)$ of non-isomorphic models of a theory T (not necessarily first order) as a function of the cardinality κ of the model. Stable FOL-theories have, roughly speaking, slow growing generalized spectra, and their models carry a kind of geometric structure. For unstable theories the generalized spectrum grows fast, and no such geometry is available. In the proof of Theorem 1.2 ultimate periodicity is achieved by showing that the models of ϕ are disjoint unions of particularily simple structures. Hence the ultimate periodicity of $spec(\phi)$ may be viewed as reflecting some structural properties of the models of ϕ. The analogue of stability then may be a necessary and sufficient model theoretic condition for the spectrum to be ultimately periodic. Contrary to the case of Asser's problem, looking at larger fragments of SOL-logic makes the problem more interesting.

In this paper we study spectra of an extension of monadic second order logic by modular counting quantifiers $C_{k,m}$, denoted by $CMSOL$. Here $C_{k,m}x\phi(x)$ is interpreted as "there are, modulo m, exactly k elements satisfying $\phi(x)$". Instead of restrictions on the vocabulary we look at restrictions on the models. Let us explain this in the case of labeled possibly directed graphs, i.e., models with one binary and finitely many unary relation symbols. This includes words, viewed as finite linear orders with unary predicates, and labeled trees. It follows from well known results in automata theory, cf. [Str94] and [GS97], that the spectrum of an $MSOL$-sentence ϕ, where all the finite models of ϕ are words or labeled trees, is ultimately periodic. In the case of words, one combines the fact that the regular languages are exactly the $MSOL$-definable sets of words, with the pumping lemma for regular languages. In the case of labeled trees, *regular* is replaced by *recognizable.*

In the eighties the notion of tree-width of a graph became a central focus of reserach in graph theory through the work of Robertson and Seymour and its algorithmic consequences. The literature is very rich, but good references and orientation may be found in [Die96, Bod93, Bod97]. Tree-width is a parameter that measures to what extent a graph is similar to a tree. Additional unary predicates do not affect the tree-width. Tree-width of directed graphs is defined as the tree-width of the underlying undirected graph[3]. Trees have tree-width 1. The clique K_n has tree-width $n-1$. It is easy to see that the models of one unary function have tree-width at most 2. Furthermore, for fixed k, the class of finite graphs of tree-width atmost k, $TW(k)$, is $MSOL$-definable. We shall give the necessary definitions in Section 3.

[3]In [JRST03] a different definition is given, which attempts to capture the specific situation of directed graphs. But the definition above is the one which is used when dealing with hypergraphs and general relational structures.

Our first result is:

Theorem 1.4. *Let ϕ be an $CMSOL$ sentence and $k \in \mathbb{N}$. Assume that all the models of ϕ are in $TW(k)$. Then spec(ϕ) is ultimately periodic.*

This generalizes Theorem 1.2. Our proof uses Courcelle's version of the Feferman-Vaught Theorem, [Cou90], and very little of the properties of $TW(k)$. All we need is that $TW(k)$ can be generated by some $eNCE$ graph grammar, cf. [Kim97].

Theorem 1.4 follows from:

Theorem 1.5. *Let K be a class of graphs which is generated by some $eNCE$-grammar and let ϕ be a $CMSOL$ sentence. Assume that all the models of ϕ are in K. Then spec(ϕ) is ultimately periodic.*

Special cases of classes of graphs generated by an $eNCE$-grammar are $TW(k)$ and the classes $CW(k)$ of graphs of clique-width at most k. The notion of clique-width was introduced in [CER93] and studied more systematically in [CO00]. Cliques are in $CW(2)$ and trees are in $CW(3)$. In [CER93] it is shown that for every k, $TW(k) \subseteq CW(2^{k+1}+1)$.

In [GM03] the following is shown:

Theorem 1.6 (Glikson and Makowsky)**.** *Let K be a class of graphs which is generated by some $eNCE$-grammar. Then there exists $k \in \mathbb{N}$ such that $K \subseteq CW(k)$.*

So Theorem 1.5 follows from the following:

Theorem 1.7. *Let ϕ be an $CMSOL(\tau)$ sentence and $k \in \mathbb{N}$. Assume that all the models of ϕ are in $CW(k)$. Then spec(ϕ) is ultimately periodic.*

The most general form of Theorem 1.7 will be given in Section 6 as Theorem 6.1. Theorem 1.7 gives also a new method to show that certain classes of graphs or relational structures are not of bounded clique-width. Previous methods for graphs only were introduced in [MR99].

The following example is noteworthy because the spectrum is easily computed and exhibits the features which are at the heart of our main theorem.

Example 1.8. *Let $Grid_{n,m}$ be the structure with four partial unary successor functions s_{north}, s_{south}, s_{west}, s_{east}, which cancel and commute* *in the obvious way whenever they are defined:*

$$s_{north}(s_{south}(x)) = s_{south}(s_{north}(x)) = x,$$
$$s_{east}(s_{west}(x)) = s_{west}(s_{east}(x)) = x,$$
$$s_{north}(s_{east}(x)) = s_{east}(s_{north}(x)),$$
$$s_{north}(s_{west}(x)) = s_{west}(s_{north}(x)),$$
$$s_{south}(s_{east}(x)) = s_{east}(s_{south}(x)),$$
$$s_{south}(s_{west}(x)) = s_{west}(s_{south}(x)).$$

The north-boundary is the set where s_{north} is not defined. Similarly for $s_{south}, s_{west}, s_{east}$.

Let $SGrid_{n,m}$ obtained from $G_{n,m}$ by identifying the west-boundary with the east-boundary pointwise, and identifyng all points of the north-boundary (south-boundary) into one point, the north pole (the south-pole). This is like a grid on a sphere. All points different from the pole have in/out degree 2. The poles have degree m.

Let $TGrid_{n,m}$ obtained from $G_{n,m}$ by identifying the west-boundary with the east-boundary, and the south-boundary with the north-boundary, pointwise. This is like a grid on a torus. All points have in/out degree 2.

We denote by Grid (SGrid, TGrid) the class of all grids (sphere grids, torus grids) with $n, m \geq 2$, and by $SGrid_r$ the grids on the sphere where the poles have exactly r-neighbors. They are all $MSOL$-definable, even as graphs where the edge relation is the symmetric closure of the union of the successor relations. Furthermore,

(i) *$SGrid_4$ is of tree-width at most 8 with spectrum $\{4n + 2 : n \in \mathbb{N} - \{0, 1\}\}$. This is ultimately periodic.*
(ii) *TGrid is of unbounded clique-width with spectrum $\{mn : m, n \in \mathbb{N} - \{0, 1\}\}$. This is not ultimately periodic.*

R. Parikh's celebrated theorem, first proved in [Par66], counts the number of occurences of letters in k-letter words of context-free languages. For a given word w, the numbers of these occurences is denoted by a vector $n(w) \in \mathbb{N}^k$, and the theorem states

Theorem 1.9 (Parikh 1966)**.** *For a context-free language L, the set $Par(L) = \{n(w) \in \mathbb{N}^k : w \in L\}$ is semilinear.*

Detailed definitions of semilinear sets[4] and related concepts are given in Section 7.

B. Courcelle has generalized this further to context-free vertex replacement graph grammars, [Cou95]. Our Theorems 7.4 and 7.15. give

[4]The terminology is from [Par66], and has since become standard terminology in formal language theory.

further generalizations of Parikh's Theorem. Rather than counting occurences of letters we look at many-sorted structures and the sizes of the different sorts, which we call many-sorted spectra. We prove that

Theorem 1.10. *Let K be a class of $CMSOL$-definable many-sorted relational structures which are of patch-width at most k. Then the many-sorted spectrum of K forms a semilinear set.*

In [FM03a], the relative strength of the weaker assumption that all the structures are of patch-width at most k is discussed in detail. Here we just note that, for relational structures, there are classes of τ-structures K which are of unbounded (relational) clique-width, but of bounded patch-width.

The proofs of all the main results have two ingredients:

- Reduction
- Pumping

If we wanted to prove the theorems only for graphs of bounded clique-width, the reduction part of the proof could be shortened by using a theorem due to B. Courcelle, [Cou95, Theorem 3.2]. On the other hand, using Courcelle's Theorem would require more prerequisites on graph grammars, which our proof avoids.

Outline of the paper. The paper is organized as follows. In Section 2 we give the necessary background for the logic $CMSOL$. In Section 3 we define tree-width, clique-width and patch-width. In Section 4 we prove a reduction from finite models of a $CMSOL$-sentence ϕ to labeled finite trees satisfying some $MSOL$-sentence ψ. In Section 5 we state the Pumping Lemma for $MSOL$-definable classes of labeled trees, In Section 6 we prove Theorem 6.1, from which all the others follow, and give some applications. In Section 7 we look at many-sorted spectra, i.e spectra of definable classes of many-sorted structures. This allows us to extend the results to Guarded Second Order Logic $GSOL$, introduced first in [Mak99, GHO00], provided the the models are all of tree-width at most k. In Section 8, finally, we discuss further research and how our results compare to recent unpublished work of S. Shelah [She04].

2. The logic $CMSOL$

We assume the reader is familiar with basic finite model theory and descriptive complexity theory as described in, say, [EFT80, EF95, Imm99, Str94].

2.1. **General background.** A vocabulary τ is a finite set of relation symbols, function symbols and constants. $FOL^q(\tau)$ and $MSOL^q(\tau)$ denote the set of τ-formulas in first order logic, respectively monadic second order logic, of quantifier rank at most q. For the definition of quantifier rank we do not distinguish between first order or second order quantification. If q is omitted, we mean all formulas. A *sentence* is a formula without free variables. For a class of τ-structures K, $Th^q_{FOL}(K)$ is the set of sentences of $FOL^q(\tau)$ true in all $\mathfrak{A} \in K$. We write $Th^q_{FOL}(\mathfrak{A})$ for $K = \{\mathfrak{A}\}$. Similarly, $Th^q_{MSOL}(K)$ denotes the corresponding sets of sentences for $MSOL$. For a set of sentences $\Sigma \subseteq MSOL(\tau)$ we denote by $Mod(\Sigma)$ the class of τ-structures which are models of Σ.

We treat free variables as *uninterpreted constants*. In particular, we tacitly assume that whenever we write $\phi(\bar{x}, \bar{U}) \in MSOL(\tau)$ we think of $\phi(\bar{a}, \bar{P}) \in MSOL(\tau \cup \{\bar{a}, \bar{P}\})$ where $\bar{a}$ are the uninterpreted constants corresponding to $\bar{x}$ and $\bar{P}$ are the uninterpreted unary relation symbols corresponding to $\bar{U}$. This allows us to speak of theories with free variables without having to deal with the free variables separately.

2.2. **Modular counting quantifiers.** We now add to the inductive definition of $MSOL$ the quantifiers $C_{k,m}$ where $k, m \in \mathbb{N}$ and $C_{k,m}x\phi(x)$ is interpreted as "there are, modulo m, exactly k elements x satisfying $\phi(x)$". This gives us the logic $CMSOL$.

The notion of quantifier rank $qr(\phi)$ of a formula ϕ extends naturally. For technical reasons we give the quantifier $C_{k,m}$ rank m. $CMSOL^q(\tau)$ denotes the set of $CMSOL$-formulas of quantifier rank at most q. For a class of τ-structures K, $Th^q_{CMSOL}(K)$ is the set of sentences of $CMSOL^q(\tau)$ true in all $\mathfrak{A} \in K$. A set $\sigma \subseteq CMSOL^q(\tau)$ is a *q-complete theory* if it is logically equivalent to $Th^q_{CMSOL}(\mathfrak{A})$ for some finite τ-structure $\mathfrak{A}$.

The following is folklore. It forms the basis of our argument in Section 4.

Lemma 2.1. *Up to logical equivalence, $CMSOL^q(\tau)$ is finite and there are only finite many q-complete theories.*

Proof. Here we use the fact that the quantifier rank $qr(C_{k,m}\phi)$ of $C_{k,m}\phi$ is defined as $qr(\phi) + m$. The rest is standard, cf. [EF95]. □

2.3. **Expressive power.** We look at graphs $G = \langle V, E\rangle$ as τ^1_{graph}-structures. The edge relation E is the interpretation of binary relation symbol R_E, $\tau^1_{graph} = \{R_E\}$. The cardinality of G is the cardinality of V. Sometimes graphs are made into τ^2_{graph}-structures U, V, E, S with universe $U = V \sqcup E$, two unary relation V and E and a ternary incidence relation S, hence $\tau^2_{graph} = \{P_V, P_E, R_S\}$. The cardinality of such a graph is the sum of the cardinalities of V and E, which is unnatural for the spectrum problem. But in Section 7 we shall look at this case more closely.

All the non-definability statements in the examples below can be proved using Ehrenfeucht-Fraïssé games. The definability statements are straightforward.

Example 2.2. *Typical graph theoretic concepts expressible in FOL are*

(i) *The presence or absence (up to isomorphism) of a fixed subgraph H.*
(ii) *The presence or absence (up to isomorphism) of a fixed induced subgraph H.*
(iii) *fixed lower or upper bounds on the degree of the vertices (hence also r-regularity).*

Example 2.3. *Typical graph theoretic concepts expressible in $MSOL$ but not in FOL are*

(i) *Connectivity, k-connectivity, reachability.*
(ii) *k-colorability (of the vertices).*
(iii) *The classes of grids $Grids, SGrids, TGrids$, when considered as simple graphs.*
(iv) *The presence or absence of a fixed topological minor. This includes planarity.*
(v) *The presence or absence of a fixed minor. This includes planarity. and more generally, graphs of a fixed genus g.*

Example 2.4. *Typical graph theoretic concepts expressible in $CMSOL$ but not in $MSOL$ are*

(i) *The existence of an Eulerian circuit (path),*
(ii) *The size of a connected component is a multiple of k.*
(iii) *The number of connected components is a multiple of k.*

Example 2.5. *The following are not $MSOL$-definable classes of graphs:*

(i) *The existence of a Hamiltonian circuit or path is not definable in $CMSOL(\tau^1_{graph})$, but it is in $MSOL(\tau^2_{graph})$.*
(ii) *The class of partial grids, i.e., spanning subgraphs of the grids $Grid$, is not $MSOL(\tau^1_{graph})$-definable, and not even $CMSOL(\tau^2_{graph})$-definable, cf.* [Rot98].

2.4. **Complete theories of disjoint unions.** The disjoint union of two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is denoted by $\mathfrak{A} \sqcup \mathfrak{B}$.

In [Cou90], Courcelle showed that an analogue of a Theorem of Beth, [Bet54][5] holds for $CMSOL$.

Theorem 2.6 (Courcelle 1990)**.**
For every $q \in \mathbb{N}$ and every sentence $\phi \in CMSOL^q(\tau)$ one can compute in polynomial time in the size of ϕ *a sequence of sentences*

$$\langle \psi^A_1, \ldots, \psi^A_m, \psi^B_1, \ldots, \psi^B_m \rangle \in CMSOL^q(\tau)^{2m}$$

[5]This is a very special case of the Feferman-Vaught Theorem from [FV59], and its generalizations by Gurevich and Shelah, cf. [She75, Gur79] . But this special case suffices for our applications. For a history of the precursors of the Feferman-Vaught Theorem, and its algorithmic applications , cf. [Mak01].

and a boolean function $B_\phi : \{0,1\}^{2m} \rightarrow \{0,1\}$ *such that*

$$\mathfrak{A} \sqcup \mathfrak{B} \models \phi$$

if and only if

$$B_\phi(b_1^A, \ldots b_m^A, b_1^B, \ldots b_m^B) = 1$$

where $b_j^A = 1$ *iff* $\mathfrak{A} \models \psi_j^A$ *and* $b_j^B = 1$ *iff* $\mathfrak{B} \models \psi_j^B$.

A detailed proof is found in [Cou90, Lemma 4.5, page 46ff].

From Theorem 2.6 we get

Corollary 2.7. *Let* $\mathfrak{A}$ $\mathfrak{A}'$ *and* $\mathfrak{B}$ $\mathfrak{B}'$ *be* τ*-structures such that* $Th^q_{CMSOL}(\mathfrak{A}) = Th^q_{CMSOL}(\mathfrak{A}')$ *and* $Th^q_{CMSOL}(\mathfrak{B}) = Th^q_{CMSOL}(\mathfrak{B}')$. *Then* $Th^q_{CMSOL}(\mathfrak{A} \sqcup \mathfrak{B}) = Th^q_{CMSOL}(\mathfrak{A}' \sqcup \mathfrak{B}')$

3. Tree-Width, Clique-Width and Patch-width

Here we define the notions of tree-width and clique-width, and a further generalization, the patch-width[6]. The reader not so familiar with graph theory may consult the encyclopedic [BLS99] for the terminology used in the examples. However, to understand our main result, this is not needed.

3.1. **Tree-width.**

For a survey on tree-width see [Bod98] or [DF99].

Definition 3.1 (Tree-width)**.** *A* k*-tree decomposition of a graph* $G = (V, E)$ *is a pair* $(\{X_i \mid i \in I\}, T = (I, F))$ *with* $\{X_i \mid i \in I\}$ *a family of subsets of* V*, one for each node of* T*, and* T *a tree such that*

(i) $\bigcup_{i \in I} X_i = V$.
(ii) *for all edges* $(v, w) \in E$ *there exists an* $i \in I$ *with* $v \in X_i$ *and* $w \in X_i$.
(iii) *for all* $i, j, k \in I$*: if* j *is on the path from* i *to* k *in* T*, then* $X_i \cap X_k \subseteq X_j$.
(iv) *for all* $i \in I$, $|X_i| \leq k + 1$.

A graph G *is of* tree-width at most k *if there exists a* k*-tree decomposition of* G*. A class of graphs* K *is a* $TW(k)$*-class iff all its members have tree width at most* k.

Given a graph G and $k \in \mathbb{N}$ there are efficient algorithms which determine whether G has tree-width k, and if the answer is yes, produce a tree decomposition, cf. [Bod97].

We can easily modify this definition for relational structures by that (ii) in the above definition is replaced by

(ii-rel): For each r-ary relation R, if $\bar{v} \in R$, there exists an $i \in I$ with $\bar{v} \in X_i^r$.

Example 3.2. *The following graph classes are of tree-width at most* k*:*

[6]The second author has defined this some time ago, and it appears implicitely in [CMR01] and [CM02]. A slightly more general notion appears in [Mak01].

(i) *Planar graphs of radius r with $k = 3r$.*
(ii) *Chordal graphs with maximal clique of size c with $k = c - 1$.*
(iii) *Interval graphs with maximal clique of size c with $k = c - 1$.*

Example 3.3. *The following graph classes have unbounded tree-width and are all $MSOL$-definable.*

(i) *All planar graphs and the class of all planar grids $G_{m,n}$.*
Note that if $n \leq n_0$ for some fixed $n_0 \in \mathbb{N}$, then the tree-width of the grids $G_{m,n}, n \leq n_0$, is bounded by $2n_0$.
(ii) *The regular graphs of degree 4 have unbounded tree-width.*
To see this note that the grids $Grid, SGrid, TGrid$ considered as simple graphs, have unbounded tree-width, but the grids in $SGrid_4$ have bounded tree-width.

3.2. **Clique-width.** A k-coloured τ-structures is a $\tau_k = \tau \cup \{P_1, \ldots, P_k\}$-structure where $P_i, i \leq k$ are unary predicate symbols the interpretation of which are disjoint (but can be empty).

Definition 3.4. *Let $\mathfrak{A}$ be a k-coloured τ-structure.*

(i) *(Adding hyper-edges) Let $R \in \tau$ be an r-ary relation symbol. $\eta_{R,P_{j_1},\ldots,P_{j_r}}(\mathfrak{A})$ denotes the k-coloured τ structure $\mathfrak{B}$ with the same universe as $\mathfrak{A}$, and for each $S \in \tau_k$, $S \neq R$ the interpretation is also unchanged. Only for R we put*

$$R^B = R^A \cup \{\bar{a} \in A^r : a_i \in P^A_{j_i}\}.$$

We call the operation η hyper edge creation*, or simply* edge creation *in the case of directed graphs. In the case of undirected graphs we denote by $\eta_{P_{j_1},P_{j_2}}$ the operation of adding the corresponding undirected edges.*
(ii) *(Recolouring) $\rho_{i,j}(\mathfrak{A})$ denotes the k-coloured τ structure $\mathfrak{B}$ with the same universe as $\mathfrak{A}$, and all the relations unchanged but for P^A_i and P^A_j. We put*

$$P^B_i = \emptyset \text{ and } P^B_j = P^A_j \cup P^A_i.$$

We call this operation recolouring.
(iii) *(modification via quantifier free translation) More generally, for $S \in \tau_k$ of arity r and $B(x_1, \ldots, x_r)$ a quantifier free τ_k-formula, $\delta_{S,B}(\mathfrak{A})$ denotes the k-coloured τ structure $\mathfrak{B}$ with the same universe as $\mathfrak{A}$, and for each $S' \in \tau_k$, $S' \neq S$ the interpretation is also unchanged. Only for S we put*

$$S^B = \{\bar{a} \in A^r : \bar{a} \in B^A\}.$$

The operations of type ρ and η are special cases of the operation of type δ.

Definition 3.5 (Clique-Width, [CO00, Mak01])**.**

(i) *Here* $\tau = \{R_E\}$ *is symbol for the edge relation. Given a graph* $G = (V, E)$, *the* clique-width of G (*cwd(G)) is the minimal number of colours required to obtain the given graph as an* $\{R_E\}$*-reduct from a* k*-coloured graph constructed inductively from coloured singletons and closure under the following operations:*
(i.a) *disjoint union* ($\sqcup$)
(i.b) *recolouring* ($\rho_{i\to j}$)
(i.c) *edge creation* (η_{E,P_i,P_j})
(ii) *For* τ *containing more than one binary relation symbol, we replace the edge creation by the corresponding hyper edge creation* $\eta_{R,P_{j_1},\dots,P_{j_r}}$ *for each* $R \in \tau$.
(iii) *A class of* τ*-structures is a* $CW(k)$*-class if all its members have clique-width at most* k.

Remark 3.6. *If* τ *contains a unary predicate symbol* U*, the interpretation of* U *is not affected by the operations recoloring or edge creation. Only the disjoint union affects it.*

A description of a graph or a structure using these operations is called a *clique-width parse term* (or *parse term*, if no confusion arises). Every structure of size n has clique-width at most n. The simplest class of graphs of unbounded tree-width but of clique-width at most 2 are the cliques. Given a graph G and $k \in \mathbb{N}$, determining whether G has clique-width k is in **NP**. A polynomial time algorithm was presented for $k \leq 3$ in [CHL+00]. It remains open whether for some fixed $k \geq 4$ the problem is **NP**-complete. The recogniztion problem for clique-width of relational structures has not been studied so far even for $k = 2$.

Theorem 3.7 (Courcelle and Olariu, Glikson and Makowsky)**.** *Let* K *be a* $TW(k)$*-class. Then*
(i) *If* K *is class of graphs, then* K *is a* $CW(m)$*-class with* $m \leq 2^{k+1} + 1$.
(ii) *In general,* K *is a* $CW(m')$*-class with* $m' \leq f(k)$ *for some function* $f(k) = O(2^{p(k)})$ *where* p *is a polynomial in* k.

Remark 3.8. *In contrast to* $TW(k)$*, we do not know whether the class of all* $CW(k)$*-graphs is* $MSOL$*-definable.*

The following examples are from [MR99, GR00].

Example 3.9. *The following graph classes are of clique-width at most* k*:*
(i) *The cographs with* $k = 2$.
(ii) *The distance-hereditary graphs with* $k = 3$.
(iii) *The cycles* C_n *with* $k = 4$.
(iv) *The complement graphs* $\bar{C}_n$ *of the cycles* C_n *with* $k = 4$.

The cycles C_n have tree-width at most 2, but the other examples have unbounded tree-width.

Example 3.10. *The following graph classes have unbounded clique-width:*

(i) *The class of all finite graphs.*
(ii) *The class of unit interval graphs.*
(iii) *The class of permutation graphs.*
(iv) *The regular graphs of degree 4 have unbounded clique-width. The grids $Grid, SGrid, TGrid$ considered as simple graphs, have unbounded clique-width, but, as stated before, the grids in $SGrid_4$ have bounded tree-width, hence bounded clique-width.*

For more non-trivial examples, cf. [MR99, GR00].
To find more examples it is useful to note, cf. [MM03]:

Proposition 3.11. *If a graph is of clique-width at most k and G' is an induced subgraph of G, then the clique-width of G' is at most k.*

3.3. **Patch-width.** Here is a further generalization of clique-width for which our theorem still works. The choice of operation is discussed in detail in [CM02].

Definition 3.12. *Given a τ-structure $\mathfrak{A}$, the* patch-width of G *($pwd(G)$) is the minimal number of colours required to obtain $\mathfrak{G}$ as an $\{\tau\}$-reduct from a k-coloured τ-structure inductively from fixed finite number of τ_k-strcutures and closure under the following operations:*

(i) *disjoint union ($\sqcup$),*
(ii) *recoloring ($\rho_{i\to j}$) and*
(iii) *modifications ($\delta_{S,B}$).*

A class of τ-structures is a $PW_\tau(k)$-class if all its members have patch-width at most k.

A description of a τ-structure using these operations is called a *patch term.*

Example 3.13.

(i) *In* [CO00] *it is shown that if a graph G has clique-width at most k then its complement graph $\bar{G}$ has clique-width at most $2k$. However, its patch-width is also k as $\bar{G}$ can be obtained from G by $\delta_{E,\neg E}$.*
(ii) *The clique K_n as a τ^1_{graph}-structure has clique-width 2. Considered as a τ^2_{graph}-structure it has clique-width $c(n)$ and patch-width $p(n)$ where $c(n)$ and $p(n)$ are functions which tend to infinity. This will easily follow from Theorem 7.4. For the clique-width of K_n as a τ^2_{graph}-structure this was already shown in* [Rot98].

Remark 3.14. *In* [CM02] *it is shown that a class of graphs of patch-width at most k is of clique-width at most $f(k)$ for some function f. It is shown in* [FM03a] *that this is not true for relational structures in general.*

As in the operation $\delta_{S,B}$ the formula B is quantifier free we have directly.

Lemma 3.15. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures such that $Th^k_{CMSOL}(\mathfrak{A}) = Th^k_{CMSOL}(\mathfrak{B})$. Then $Th^k_{CMSOL}(\delta_{S,B}(\mathfrak{A})) = Th^k_{CMSOL}(\delta_{S,B}(\mathfrak{B}))$.*

As there are, up to logical equivalence, only finitely many quantifier free τ-formulas with a fixed number of free variables, we get:

Lemma 3.16. *For fixed finite relational τ, there are only finitely many operations $\delta_{S,B}$.*

Remark 3.17. *In the definition of patch-width we allowed only unary predicates as auxiliary predicates (colours). We could also allow r-ary predicates and speak of r-ary patch-width. The theorems where bounded patch-width is required are also true for this more general case. The relative strength of clique-width and the various forms of patch-width are discussed in* [FM03a].

For further use we note

Lemma 3.18. *Let $\mathfrak{A}$ be a τ_k-structure with universe A and with patch-term $\mathfrak{t}(A)$*[7]*. Then the size of $\mathfrak{A}$ is the sum of the sizes of the structures which are labels of the leaves of $\mathfrak{t}(A)$.*

3.4. **Classes of unbounded patch-width.** Theorem 6.1 will give us a method to show that certain classes K of graphs have unbounded patch-width. Hence, as this is also true for every $K' \supseteq K$, the class of all graphs is of unbounded patch-width.

Without Theorem 6.1 there was only a conditional proof of unbounded patch-width available. This used the following:

(i) Checking patch-width at most k of a structure $\mathfrak{A}$, for k fixed, is in **NP**. Given a structure $\mathfrak{A}$, one just has to guess a patch-term of size polynomial in the size of $\mathfrak{A}$.

(ii) Using the results of [Mak01] one gets that checking a $CMSOL(\tau)$-property ϕ on the class $PW_\tau(k)$ is in **NP**, whereas, by [MP96], there are Σ^P_n-hard problems definable in $MSOL$ for every level Σ^P_n of the polynomial hierarchy.

(iii) Hence, if the polynomial hierarchy does not collapse to **NP**, the class of all τ-structures is of unbounded patch-width, provided τ is large enough.

[7] By abuse of notation we write for the patch-term $\mathfrak{t}(A)$ rather than $\mathfrak{t}(\mathfrak{A})$.

Problem 3.19. *What is the complexity of checking whether a τ-structure $\mathfrak{A}$ has patch-width at most k, for a fixed k?*

3.5. **Clique-width and graph grammars.** It follows from [GM03], that in the case of graph languages generated by $eNCE$-grammars, an upper bound of the clique-width of a graph can be computed in polynomial time from a derivation tree of the graph. On the one hand the upper bound obtained does not depend on the particular derivation (only on the grammar), but on the other hand, the upper bound may be far from optimal.

In [CM02] the classes of graphs generated by $C - NCE$-grammars (context-free VR-grammars) are characterized as those defined as the least solution of systems of recursive set equations based on the operations used in the definition of clique-width. Also in [CM02], based on [CE95, Cou95, Cou92, EvO97], a characterization of context-free Hyperedge Replacement grammars (HR-grammars) is given in similar terms adapted to the operations used in computing a graph from its tree decomposition (disjoint union, renaming and fusion).

4. Reduction to $MSOL$-definable classes of labeled trees

In this section we prove the main lemma needed for the proof of Theorem 1.7 and its generalizations. The lemma is a generalization of a theorem due to B. Courcelle, [Cou95, Theorem 3.2], which is phrased in terms of graph grammars, $MSOL$-definable transductions of recognizable trees, cf. also [Cou97]. Our presentation is purely model theoretic and self-contained.

Let $\phi \in CMSOL^q(\tau)$. For each τ_k-structure $\mathfrak{A}$ of patch-width at most k with patch-width parse term $\mathfrak{t}(A)$ we construct a labeled Σ-tree $t(\mathfrak{t}(A))$, where Σ depends only on τ, q and k.

Lemma 4.1 (Main Lemma). *Let $\phi \in CMSOL^q(\tau)$. There is a set of labels Σ_ϕ, and sentence $\psi \in MSOL$ over Σ_ϕ-trees, such that for every $\mathfrak{A}$ of patch-width at most k*

$$\mathfrak{A} \models \phi \text{ iff } t(\mathfrak{t}(A)) \models \psi$$

The proof has two parts: the construction of the labeling and the construction of ϕ.

4.1. **The labelings.** The parse term $\mathfrak{t}(A)$ is itself a labeled binary tree where

(i) the leaves are each labeled with one of finitely many τ_k-structures $\mathfrak{A}_a : a \in A$ for some finite set A;
(ii) the internal nodes of degree 2 are all labeld by $\sqcup$;
(iii) the internal nodes of degree 1 are labeled by one of the finite many possibilities of the versions of $\delta_b, b \in \tau \times FOL^0(\tau_k)$.
(iv) We denote the set of labels used so far by Σ_0.

By Lemma 3.16 we note that Σ_0 is finite.

Let σ_1, σ_2 be q-complete theories. So there are $\mathfrak{A}_1, \mathfrak{A}_2$ with σ_i logically equivalent to $Th^q_{CMSOL}(\mathfrak{A}_i), i = 1, 2$ respectively. We denote by $\sigma_1 \sqcup \sigma_2$ the q-complete $CMSOL$-theory of $\mathfrak{A}_1 \sqcup \mathfrak{A}_2$. This is well defined, by Corollary 2.7. We proceed similarly for $\delta_{S,B}(\mathfrak{A})$ and denote by $\delta_{S,B}(\sigma_1)$ the q-complete $CMSOL$-theory of $\delta_{S,B}(\mathfrak{A}_1)$. This is justified by Lemma 3.15.

We now add inductively new labels to $\mathfrak{t}(A)$. The set of new labels will be denoted by Σ_1. The labels $\sigma \in \Sigma_1$ are q-complete $CMSOL$-theories. Here we use Lemma 2.1. Recall that Let $\phi \in CMSOL^q(\tau)$.

(i) the leaves with Σ_0 label $\mathfrak{A}$ are each labeled with $Th^q_{CMSOL}(\mathfrak{A})$.
(ii) the internal nodes d of degree 2 where $s_1(d)$ has Σ_1-label σ_1 and $s_2(d)$ has Σ_1-label σ_2, have Σ_1-label $\sigma_1 \sqcup \sigma_2$.
(iii) the internal nodes d of degree 1 where d has Σ_0-label $\delta_{S,B}$ and $s_1(d)$ has Σ_1-label σ, have Σ_1-label $\delta_{S,B}(\sigma)$.

We put $\Sigma_\phi = \Sigma_0 \times \Sigma_1$. The labeled tree $t(\mathfrak{t}(A))$ is now the Σ_ϕ-tree obtained from $\mathfrak{t}(A)$ as defined above.

4.2. **The $MSOL$-sentence.** ψ is the conjunction of the following statements:

(i) t is a Σ_ϕ-tree.
(ii) The leaves have one of the finitely many labels $(\mathfrak{A}, \sigma_{\mathfrak{A}})$ with $\sigma_{\mathfrak{A}} = Th^q_{CMSOL}(\mathfrak{A})$.
(iii) The finite set of FOL-sentences describing inductively the labeling.
(iv) The Σ_1-label of the root is one of the σ's with $\sigma \models \phi$.

Only (i) is an $MSOL$-sentence, all the others are FOL-sentences.

Clearly every t with $t \models \psi$ is a parse tree of some structure $\mathfrak{A}$ which satisfies ϕ. This completes the proof of Lemma 4.1. □

5. The pumping lemma for $MSOL$-definable classes of trees

In this section we present a pumping lemma for $MSOL$-definable classes of trees as we need it in the sequel. We take the material from [GS97]. But we eliminate some automata theoretic terminology, namely the notion of *recognizable* sets of labeled trees. In [GS97], Proposition 12.2 states that a class of labeled trees T (viewed as relational structures) is $MSOL$-definable iff T (viewed as terms) is recognizable. But Proposition 5.2. states that a class of recognizable labeled trees T has the pumping property. So here we state the pumping property directly for binary trees viewed as relational structures.

Definition 5.1 (Labeled trees).

(i) *A labeled (binary) tree structure is a structure of the form*

$$t = \langle D, <, s_1, s_2, \{P_z : z \in \Sigma\}\rangle$$

where D is the domain of the tree, $<$ is a partial order, s_1 and s_2 are (binary) successor relations, Σ is a finite set of labels, and for every $z \in \Sigma$, P_z are disjoint unary predicates. We call these structures Σ-trees *and denote the the set of* Σ-trees *by T_Σ. The unary predicates $P_z, z \in \Sigma$ are the labels.*

(ii) *s_1 and s_2 are partial one-one functions and the partial order $<$ is the transitive closure of $s_1 \cup s_2$. Moreover, there exists no $u \neq v$ such that $s_1(u) = s_2(v)$.*

(iii) *The root is the only element which is not a successor.*

(iv) *The leaves are the elements which have no successor.*

(v) *The height of t, denoted by $hg(t)$, is defined inductively: leaves have height 0, and $hg(d) = 1+\max\{hg(s_1(d)), hg(s_2(d))\}$. This includes the case when d has only one successor.*

(vi) *t' is a subtree of t, if it is a substructure which is closed under the successor relations.*

(vii) *Every node of the tree carries a label.*

We want to define an analogue to concatenation of words for trees. The idea is to mark a distinguished leaf and attach a new tree at this leaf. We make this precise:

Definition 5.2.

(i) *Let $\xi \notin \Sigma$. A* Σ-context *is a $(\Sigma \cup \{\xi\})$-tree, where P_ξ consists of a unique leaf. We denote the set of Σ-contexts by Ct_Σ.*

(ii) *Let $p \in Ct_\Sigma \cup T_\Sigma$ and $q \in Ct_\Sigma$. We denote by $p \cdot q$ the Σ-context or Σ-tree obtained by substituting the ξ appearing in q by p. If $p \in Ct_\Sigma$ we obtain a context, if $p \in T_\Sigma$ we obtain a tree.*

(iii) *For $p \in Ct_\Sigma$ we denote by p^k the context $\underbrace{p \cdot p \cdot \ldots \cdot p}_{k}$.*

(iv) *p' is subcontext of p if it is a subtree with the same interpretation of ξ.*

As we shall use some details of the following Pumping Lemma in Section 7, we have to set up some terminology.

Definition 5.3. *Let K be a class of finite Σ-trees. A context $p \in Ct_\Sigma$ with $1 \leq hg(p)$ is a* pump for a tree $t \in K$ *if there are $s \in T_\Sigma$ and $q \in Ct_\Sigma$ such that*

(i) *$t = s \cdot p \cdot q$;*

(ii) *for every $k \in \mathbb{N}$ the tree $t' = s \cdot p^k \cdot q \in K$.*

A pump p for $T \in K$ is minimal if it does not have a subcontex p' which is also a pump for t. We denote by $MinPump(K)$ the set of minimal pumps for K.

If no ambiguity arises, we just speak of a pump for K.

Proposition 5.4 (Pumping lemma). *Let K be a class of finite Σ-trees defined by an $MSOL$-sentence ϕ.*

(i) *Then there is a number $n \in \mathbb{N}, n \geq 1$ which depends only on ϕ such that, if $t \in K$ and $hg(t) \geq n$, then t has a pump for K with $hg(p) \leq n$.*
(ii) *$MinPump(K)$ is finite.*

Proof. (i) is Proposition 5.2 from [GS97].
(ii) follows easily from (i) and the fact that the number of trees of height $\leq n$ is finite. □

We shall need in Section 7 a stronger version of the Pumping Lemma, where we have several independent pumps.

Definition 5.5. *Let K be a class of finite Σ-trees and $t \in K$. The contexts $p_1, p_2, \ldots, p_m \in Ct_\Sigma$ are* independent pumps in t for K *if there exist contexts $q_1, q_2, \ldots, q_m \in Ct_\Sigma$ and trees $s_1, s_2, \ldots, s_m \in T_\Sigma$ such that for each $i \leq m$ we have $t = s_i \cdot p_i \cdot q_i$, the vertices of the p_i's in t are pairwise disjoint, and the p_i's are simultaneous pumps for t, i.e. if t' is the tree obtained from t by replacing $p_1, p_2, \ldots, p_m$ by $p_1^{k_1}, p_2^{k_2}, \ldots, p_m^{k_m}$ simultaneously, then $t' \in K$.*

Remark 5.6. *Without making it clearer, one should really define the occurrences of the p_i's in t, possibly as multiple contexts, and then proceed with an inductive definition of simultaneous substitution.*

Theorem 5.7 (Independent Pumping Lemma). *Let K be an $MSOL$-definable class of Σ-trees. Let $t \in K$ with m independent pumps $p_1, \ldots, p_m$ in t for K. There is a number $n = n(K, p_1, \ldots, p_m)$ such that if $hg(t) \geq n$ then there is another pump p in t such that $p, p_1, \ldots, p_m$ are independent pumps in t for K.*

Proof. Same techniques as for Theorem 5.4. □

6. The main theorem and some applications

6.1. Main theorem.

Our most general theorem can now be stated:

Theorem 6.1. *Let $\phi \in CMSOL^q(\tau)$ be such that all its finite models have patch-width at most k. Then there are $m_0, n_0 \in \mathbb{N}$ such that if ϕ has a model of size $n \geq n_0$ then ϕ has also a model of size $n + m_0$.*

Proof. W.l.o.g., we can assume that ϕ has arbitrarily large models. Let K_ϕ consist of all the tree presentations $t(\mathfrak{t}(A))$ of models $\mathfrak{A}$ of ϕ. Using Lemma 4.1, all models do have tree presentations and there are Σ_ϕ-trees $t \models \psi$ of arbitrarily large height.

We now apply the Pumping Lemma (Proposition 5.4). There is a number $n_1 \in \mathbb{N}, n_1 \geq 1$ such that, if $t \models \psi$ and $hg(t) \geq n_1$, then for some $s \in T_{\Sigma_\phi}$ and $p, q \in Ct_{\Sigma_\phi}$ with $hg(p) \geq 1$ and $p \in MinPump(K_\phi)$ we have

(i) $t = s \cdot p \cdot q$,
(ii) $hg(p) \leq n_1$, and
(iii) for $t^{(k)} = s \cdot p^k \cdot q$ we have for every $k \in \mathbb{N}$ the tree $t^{(k)} \models \phi$.

Let n be the size of $\mathfrak{A}$ with $t = t(\mathfrak{t}(A))$. Let $m(p)$ be the sum of the sizes of the structures which are labels of the leaves of p, i.e., all the leaves but the one labeled ξ.

Let $\mathfrak{B}^{\mathfrak{k}}$ with $t^{(k)} = t(\mathfrak{t}(B^k))$. Then the size of $\mathfrak{B}^{\mathfrak{k}}$ is $n + (k-1)m(p)$.

To complete the proof, let m_0 be the least common multiplier of the finite set of numbers $\{m(p) : p \in MinPump(K_\phi)\}$. □

From this we get immediately:

Corollary 6.2. *Let $\phi \in CMSOL^q(\tau)$ be such that all its finite models have patch-width at most k. Then $spec(\phi)$ is ultimately periodic.*

Theorems 1.4 and 1.7 now follow immediately, as from the parse terms for tree-width or clique-width we can get the parse terms for patch-width, cf. [CO00, GM03].

6.2. **Applications.** From Theorem 6.1 we get immediately a criterion for classes of structures to have unbounded patch-width.

Corollary 6.3. *Let ϕ be $FOL(\tau)$-sentence with a non-linear polynomial spectrum. Then $Mod(\phi)$ is of unbounded patch-width (resp. clique-width, tree-width).*

We want to use this to show that indeed most first order spectra cannot be represented by sentences with models of bounded patch-width. This is of interest, as it represents so far the only methods of proving that a class K contains structures of unbounded patch-width.

Definition 6.4. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function. A spectrum is an f-spectrum if it is of the form $\{f(n) : n \in \mathbb{N}\}$. A spectrum is polynomial if it is a g-spectrum for some polynomial $g \in \mathbb{Z}[x]$ with all its values in $\mathbb{N}$.*

Clearly, a polynomial spectrum is ultimately periodic iff g is a linear function.

Example 6.5.

(i) *Let $\phi_{sq}(\tau)$ be with relation symbols $\tau = \{U, S\}$, U unary and S ternary. Let ϕ_{sq} say in a structure $\mathfrak{A}$ that S^A is a bijection between $(U^A)^2$ and $A - U^A$. Then the spectrum of ϕ_{sq} is an f-spectrum with $f(n) = n^2 + n$.*
(ii) *The FOL-sentence axiomatizing fields of characteristic p has an f-spectrum with p^{n+1}.*

In [Mor94] it is shown:

Theorem 6.6 (M. More, 1994). *Let $g \in \mathbb{Z}[x]$ with all its values in $\mathbb{N}$. Then there is an FOL-sentence ϕ_g with spectrum $spec(\phi_g) = \{g(n) : n \in \mathbb{N}\}$.*

Corollary 6.7. *For each $g \in \mathbb{Z}[x]$ with all its values in $\mathbb{N}$, the sentence ϕ_g of Theorem 6.6 has models of arbitrarily large patch-width.*

7. Many-sorted spectra

In this section we want to analyze spectra of many-sorted structures. Our motivation stems from the representation of graphs as two-sorted structures with vertices and edges as elements and an incidence relation. The vocabulary corresponding to this was denoted in Section 2.3 by τ^2_{graph}.

7.1. **Many-sorted structures.** Let $s \in \mathbb{N}$. An s-sorted vocabulary τ is a relational vocabulary which contains s unary relation symbols $U_1, \ldots U_s$. The $U_i : i \leq s$ are called *sort predicate symbols.* To simplify notation we represent s-sorted τ-structures $\mathfrak{A}$ as structures with one universe A, s unary (sort)-predicates $U_1^A, \ldots U_s^A$ with $\bigcup_{i=1}^s U_i^A = A$ and for each $i \neq j$ $U_i^A \cap U_j^A = \emptyset$. As $A \neq \emptyset$ at least one U_i has to be non-empty. $\mathfrak{A}$ is finite if A is finite. A structure is many-sorted if it is s-sorted for some $s \geq 2$. The size $msize_s(\mathfrak{A})$ of an s-sorted structure is the vector $(\mid U_1^A \mid, \mid U_2^A \mid, \ldots, \mid U_s^A \mid)$.

k-coloured many-sorted structures have additionally k unary relation symbols which are different from the sort predicate symbols. The definition of tree-width, clique-width, and patch-width can now be applied verbatim.

7.2. **Many-sorted spectra.**

Definition 7.1. (i) *Let $\mathfrak{A}$ be a finite s-sorted structure. The many-sorted size $msize(\mathfrak{A})$ is the s-vector $\bar{n} = (n_1, \ldots n_2)$ with $n_i = \mid U_i^A \mid$.*

(ii) *The s-sorted spectrum of a τ-sentence ϕ is the set of s-tuples*

$$mspec_s(\phi) = \{\bar{n} \in \mathbb{N}^s : \text{ there is } \mathfrak{A} \models \phi \text{ with } msize(\mathfrak{A}) = \bar{n}\}$$

(iii) *For $j \leq s$ we denote by $spec_j(\phi)$ the set*

$$spec_j(\phi) = \{n \in \mathbb{N} : \text{ there is } \mathfrak{A} \models \phi \text{ with } \mid U_j^A \mid = n\}$$

(iv) *A set $X \subseteq \mathbb{N}^s$ is an* arithmetic ray in $\mathbb{N}^s$ *if there are $\bar{a}, \bar{b} \in \mathbb{N}^s$ with*

$$X = A_{\bar{a},\bar{b}} = \{(a_1 + k \cdot b_1, \ldots, a_s + k \cdot b_s) \in \mathbb{N}^s : k \in \mathbb{N}\}$$

Singletons are arithmetic rays with $\bar{b} = \bar{0}$. If $\bar{b} \neq \bar{0}$ the ray is a proper ray.

(v) *A set* $X \subseteq \mathbb{N}^s$ *is* linear in $\mathbb{N}^s$ *iff there is vector* $\bar{a} \in \mathbb{N}^s$ *and a matrix* $M \in \mathbb{N}^{s \times r}$ *such that*

$$X = A_{\bar{a},\bar{M}} = \{\bar{b} \in \mathbb{N}^s : \text{ there is } \bar{u} \in \mathbb{N}^r \text{ with } \bar{b} = \bar{a} + M \cdot \bar{u}\}$$

Singletons are linear sets with $M = 0$*. If* $\bar{M} \neq 0$ *the series is* nontrivial.

(vi) $X \subseteq \mathbb{N}^s$ *is* semilinear in $\mathbb{N}^s$ *iff* X *is a finite union of linear sets* $A_i \subseteq \mathbb{N}^s$.

Example 7.2. *For* $p \in \mathbb{N}$ *the set*

$$X_p = \{(m,n) : \text{ there is } k \in \mathbb{N} \text{ with } m = k \cdot p, m \leq n\}$$

is a countable union of proper arithmetic rays and also a linear set. Note that every linear set is a countable union of arithmetic rays, but not conversely.

Inspecting the proof of Theorem 6.1 and using Remark 3.6 we get immediately:

Proposition 7.3. *Let* τ *be an* s*-sorted vocabulary and* $\phi \in CMSOL(\tau)$ *with all its models of patch-width at most* k*. Then the many-sorted spectrum* $mspec_s(\phi)$ *is a countable union of proper arithmetic rays and a finite number of singletons.*

To get the following characterization one has to work a bit harder.

Theorem 7.4. *Let* τ *be an* s*-sorted vocabulary and* $\phi \in CMSOL(\tau)$ *with all its models of patch-width at most* k*. Then the many-sorted spectrum* $mspec_s(\phi)$ *is a semilinear set.*
In particular, for every $s' \subseteq s$*, all the spectra* $spec_{s'}(\phi)$ *are semilinear sets in* $\mathbb{N}^{s'}$.

Proof. We use Lemma 4.1. Let ϕ the $MSOL$-sentence over Σ_ϕ-trees encoding the models of ϕ, and let $K = Mod(\phi)$. $MinPump(K)$ is finite by Lemma 5.4(ii). Let $\mathcal{A}$ be the finite set of structures with $t(\mathfrak{t}(A)) \in MinPump(K)$. Finally, let

$$P = \{\bar{p} \in \mathbb{N}^s : \bar{p} = size_s(\mathfrak{B}) \text{ and } t(\mathfrak{t}(B)) \in MinPump(K)\}$$

with $\mid P \mid = r_P$. Let X be the s-spectrum of ϕ. Let $\mathcal{X}$ be the set of maximal linear sets $Y \subseteq X$ of the form $A_{\bar{a},M}$, defined by $\bar{a} \in \mathbb{N}^s$ and $M \in \mathbb{N}^{s \times r'}$, with $r' \leq r_P$, which result from pumping in a single structure the independendent pumps $p_1, \ldots, p_r$ corresponding to the column vectors of M, which are from $P \cup \{\bar{0}\}$.

Claim 1. *There are only finitely many such matrices* M.

Define $\mathcal{X}_M \subseteq \mathcal{X}$ by

$$\{Y \in \mathcal{X} : \text{ there is } \bar{a} \in \mathbb{N}^s \text{ such that } Y = A_{\bar{a},M}\}$$

Obviously we have

Claim 2. *$X = \bigcup \mathcal{X}$.*

Claim 3. *$\mathcal{X}$ is finite.*

Assume otherwise. Using Claim 1, we conclude that for some M also $\mathcal{X}_M$ is infinite. Using Theorem 5.7, there is some $\mathfrak{A} \models \phi$ such that $size_s(\mathfrak{A}) = \bar{a}$ with $A_{\bar{a},M} = \mathcal{X}_M$ such that $t = t(\mathfrak{t}(A)) = s' \cdot p \cdot q'$ has a pump p, such that $p, p_1, \ldots, p_r$ are independent pumps in t for K.

Let $\mathfrak{B}$ be a structure such that $t(\mathfrak{t}(B)) = p$. Denote by $\bar{p} = size_s(\mathfrak{B})$

Case 1. *$\bar{p}$ is a linear combination of column vectors of M with coefficients in $\mathbb{N}$.*

Let $\mathfrak{A}'$ be the structure such that

$$t(\mathfrak{t}(A')) = t' = s' \cdot p^0 \cdot q' = s' \cdot q'$$

and $\bar{a}' = size_s(\mathfrak{A}')$. Then $A_{\bar{a},M}$ is a proper subset of $A_{\bar{a}',M}$, which contradicts the maximality of $A_{\bar{a},M}$.

Case 2. *$\bar{p}$ is not a linear combination of column vectors of M with coeeficients in $\mathbb{N}$.*

Then $r' < r_P$, Let M' be the matrix obtained from M by adding $\bar{p}$ as a new column to M. Then $A_{\bar{a},M}$ is a proper subset of $A_{\bar{a},M'}$, which contradicts the maximality of $A_{\bar{a},M}$. □

The converse is also true, even for FOL-definable classes where all the models are of bounded tree-width. As we have in general $spec_s(\phi) \cup spec_s(\psi) = spec_s(\phi \vee \psi)$ it suffices to show that every linear set is a spectrum.

Theorem 7.5. *Let $k, s \in \mathbb{N}$ and $\bar{a} \in \mathbb{N}^s, M \in \mathbb{N}^{s \times r}$, and M linear. There is an s-sorted $FOL(\tau)$-sentence ϕ such that*

(i) *All sufficiently large models of ϕ of have tree-width exactly k.*
(ii) *$spec_s(\phi) = A_{\bar{a},M}$*

This follows immediately from the following

Lemma 7.6. *Let $G_0, G_1, \ldots, G_t$ be a set of pairwise distinct connected s-sorted graphs (that is, s-sorted structures with one binary edge relation). Let K be the class of s-sorted graphs whose universe consists of exactly one copy of G_0, and of any number of disjoint copies of graphs from $G_1, \ldots, G_t$. Then K is FOL-definable.*

Proof. Let $g(j)$ be the size of $V(G_j)$ for $0 \leq j \leq t$, and set $g = \max_{0 \leq j \leq t} g(j)$. For any s-sorted graph G with n vertices, let $\phi_G(v_1, \ldots, v_n)$ be an FOL-formula which says that the s-sorted graph induced by the vertices $\bar{v} = (v_1, \ldots, v_n)$ is isomorphic to G. We also let $\theta_n(v_1, \ldots, v_n)$ be the FOL sentence stating that $\{v_1, \ldots, v_n\}$ is a connected component (the sizes of both ϕ_G and θ_n depend on the number of variables). Our FOL formula is the conjunction of the following.

(i) There exist no connected components of size greater than g; this is definable by

$$\forall v_1 \exists v_2, \ldots, v_g \psi(v_1, \ldots, v_g),$$

where ψ is the disjunction of all possible θ_n statements on subsets of $v_1, \ldots, v_g$ which include v_1.

(ii) Every connected component of size g or less is isomorphic to one of $G_0, \ldots, G_t$; this is definable by forbidding any connected component of size $1 \leq j \leq g$ for $j \notin \{g(0), \ldots, g(t)\}$, and for $j \in \{g(0), \ldots, g(t)\}$ stating that $\theta_j(v_1, \ldots, v_j)$ implies that the spanned induced subgraph is one of the appropriate G_i (with a disjunction of the corresponding ϕ_{G_i} predicates).

(iii) There exist $v_1, \ldots, v_{g(0)}$ such that

$$\theta_{g(0)}(v_1, \ldots, v_{g(0)}) \wedge \phi_{G_0}(v_1, \ldots, v_{g(0)}).$$

(iv) There exist no $v_1, \ldots, v_{2g(0)}$ for which

$$\theta_{g(0)}(v_1, \ldots, v_{g(0)}) \wedge \phi_{G_0}(v_1, \ldots, v_{g(0)}) \wedge$$
$$\theta_{g(0)}(v_{g(0)+1}, \ldots, v_{2g(0)}) \wedge \phi_{G_0}(v_{g(0)+1}, \ldots, v_{2g(0)}).$$

□

Proof of Theorem 7.5. We let $G_1, \ldots, G_r$ each be an s-sorted tree, for which the number of vertices of each sort corresponds to a column of A. If $\bar{a}$ is sufficiently large in terms of the sum of its coordinates, then we let G_0 be any s-sorted graph of tree-width exactly k, and use the class K of the lemma above to obtain our first order sentence, in which the number of vertices of each sort corresponds to $\bar{a}$.

If $\bar{a}$ is not large enough, then it is easy find $\bar{a}_1, \ldots, \bar{a}_l$ such that each of them is large enough to admit a corresponding s-sorted graph of tree-width k, and satisfying also that $\bigcup_{i=1}^{l} A_{\bar{a}_i,M} \subset A_{\bar{a}_i,M}$, and that $A_{\bar{a}_i,M} \setminus \bigcup_{i=1}^{l} A_{\bar{a}_i,M}$ is finite. We then let our first order sentence be the conjunction of the sentences for $A_{\bar{a}_i,M}$, with explicit descriptions of the finite number of models corresponding to $A_{\bar{a}_i,M} \setminus \bigcup_{i=1}^{l} A_{\bar{a}_i,M}$. □

7.3. **Many-sorted spectra and Parikh's Theorem.** Spectra of many-sorted structures are similar to Parikh Mappings in the case of words, cf. [Hab92, Chapter IV.4]. Parikh Mappings associate with each word w over an alphabet $\Sigma = \{a_1, \ldots, a_s\}$ with s letters a vector $n(w) = (n_1(w), \ldots n_s(w)) \in \mathbb{N}^s$ where $n_i(w)$ denotes the number of occurences of a_i in w. For a language $L \in \Sigma^*$, we define $Par(L) = \{n(w) : w \in L\}$. This definition is easily adapted to vertex-labeled graphs.

Parikh's Theorem (Theorem 1.9) from the introduction asserts that for a context-free language L, the set $Par(L)$ is semilinear. A. Habel generalized Parikh's Theorem to hyperedge-replacement graph languages, [Hab92], and B. Courcelle extended this to context-free vertex-replacement graph languages, [Cou95]. Our Theorem 7.4 can be viewed as another variation of Parikh's Theorem, as stated in the introduction as Theorem 1.10.

7.4. **Application to graph theory.** Here we want to compare spectra of graphs viewed as τ^1_{graph} and as τ^2_{graph}-structures[8]. We call the many-sorted spectrum here the *vertex-edge spectrum*, and the spectra of the particular sorts *vertex-spectrum* and *edge-spectrum* respectively. From Courcelle's and our work we know the following, cf. [CMR01]:

Let $G = \langle V, E\rangle$ be a graph viewed as a τ^1_{graph}-structure. Let $tr(G) = \mathfrak{A}$ be the τ^2_{graph}-structure with universe $U^A = V \sqcup E$, $P^A_V = V$, $P^A_E = E$ and $R^A = \{(u, e, v) \in V \times E \times V : e = (u, v) \in E\}$. Let K_1 be a class of τ^1_{graph}-structures and let $K_2 = \{tr(G) : G \in K_1\}$ the corresponding class of τ^2_{graph}-structures.

Proposition 7.7.

(i) *If K_1 is $MSOL$-definable ($CMSOL$-definable), so is K_2.*
(ii) *The class of Hamiltonian graphs is $MSOL$-definable as a class of τ^2_{graph}-structures, but not $CMSOL$-definable as a class of τ^1_{graph}-structures.*
(iii) *If K_1 is of bounded tree-width, so is K_2.*
(iv) *The class of cliques is of clique-width at most 2 as a class of τ^1_{graph}-structures, but of unbounded patch-width as a class of τ^2_{graph}-structures, cf. Example 6.5.*

Hence we get from Theorem 7.4 and Proposition 7.7

Corollary 7.8. *Let K be a class of graphs as τ^2_{graph}-structures defined by $\phi \in CMSOL(\tau^2_{graph})$ which is of bounded patch-width. Then the spectra $mspec_{ve}(\phi), spec_v(\phi), spec_e(\phi)$ are semilinear, respectively ultimately periodic.*

7.5. **Guarded Second Order Logic.** As we have seen concerning the difference between τ^1_{graph} and τ^2_{graph}, $MSOL$ as a logic is very sensitive to the choice of representations of graphs and the choice of vocabulary in general, cf. also [Cou97]. A more stable version is Bounded Second Order Logic $BSOL$, introduced in [Mak99], which also appears under the name of Guarded Second Order Logic $GSOL$ in [GHO00]. Here we can quantify over *subsets of the basic relations* as well. More precisely, $GSOL$ ($CGSOL$) is obtained from $MSOL$ ($CMSOL$) by adding in the inductive definition of formulas the clause:

[8] B. Courcelle has studied the difference in the expressive power of $MSOL$ for these two vocabularies in [Cou94].

For $R \in \tau$ and S a second order variable of the same arity as R, $\exists S \subseteq R\phi$ is an $CGSOL(\tau)$-formula whenever ϕ is an $CGSOL(\tau)$-formula.

By changing the vocabulary such that tuples of the old relations become elements of the new structure and by introducing many binary relations for the projections of these tuples we can reduce $GSOL$ to $MSOL$. In the case of the binary edge relation of graphs $G = \langle V, E \rangle$ this corresponds to the passage from G to the incidence graph $I(G)$.

Definition 7.9. *The* incidence graph *of* $G = \langle V, E \rangle$ *is a bipartite graph* $I(G) = \langle I(V), I(E) \rangle$ *of edges and vertices of* G *with* $I(V) = V \cup E$ *and for* $e = (v, w)$ $(v, w) \in E$ *iff* $(v, e) \in I(E)$ *and* $(w, e) \in I(E)$. *In other words we replace every edge in* E *by a path of length* 2.

Remark 7.10. *If we replace the binary edge relation* E_I^G *of* $I(G)$ *by the ternary relation relation* R^G *defined by* $(u, e, w) \in R^G$ *iff both* $(u, e), (w, e) \in E_I^G$, *we get exactly the passage from* tau^1_{graph} *-structures to* tau^2_{graph},

Furthermore, it seems to be folklore[9] that

Proposition 7.11. *If* G *is an undirected graph and has tree-width at most* k, *then its incidence graph* $I(G)$ *has also tree-width at most* k.

Proof. We take a k-tree decompostion of G. where the nodes of the tree are the subsets V_t of $V(G)$. Choose for each edge e of G the uppermost node in the tree decomposition where the edge e occurs first. To this node we attach a new son V_e which contains three vertices of $I(G)$, the edge e and its two end points. For $k \geq 2$ this gives k-tree decomposition of $I(G)$. If $k = 1$, G is a forest, and so is $I(G)$. Hence $I(G)$ also has a 1-tree decomposition. □

Remark 7.12. *Using Theorem 7.4 we see that the patch-width of the incidence graphs of cliques is unbounded, because the edge spectrum grows quadratically. But the clique-width of the cliques is* 2. *In fact, in general, the clique-width* $cw(I(G))$ *of* $I(G)$ *is not bounded by a function of* $cw(G)$.

For arbitrary relational structures we proceed as follows:

Definition 7.13. *Let* $\tau\{R_1, \ldots, R_m\}$ *be a vocabulary and denote by* $\rho(R_i)$ *the arity of* $R_i \in \tau$. *Denote by* $\tau_I\{S_1, \ldots, S_m\}$ *the vocabulary obtained from* τ *with* $\rho(S_i) = \rho(R_i) + 1$. *The* incidence structure *of a* τ*-structure* $\mathfrak{A}$ *is a two-sorted* τ_I*-structure* $I(\mathfrak{A}) = \langle I(A), P_A^{I(A)}, P_{rel}^{I(A)}, S_i^{I(A)} : i \leq m \rangle$ *of hyperedges and vertices with* $I(A) = A \sqcup \bigsqcup_{i \leq m} A^{\rho(R_i)}$, $P_A^{I(A)} = A$, $P_{rel}^{I(A)} = \bigsqcup_{i \leq m} A^{\rho(R_i)}$, *and for* $e \in P_{rel}^{I(A)}$ *we have* $e = \bar{a} \in R_i^A$ *iff* $(\bar{a}, e) \in S_i^{I(A)}$.

[9] As was pointed out by the referee.

Using this definition we get easily:

Proposition 7.14. (i) *For every formula $\phi \in CGSOL(\tau)$ one can compute a formula $tr(\phi) \in MSOL(\tau_I)$ such that*

$$\mathfrak{A} \models \phi \text{ iff } I(\mathfrak{A}) \models tr(\phi)$$

(ii) *If the tree-width of $\mathfrak{A}$ is at most k, then also the tree-width of $I(\mathfrak{A})$ is at most k.*

We note that a similar argument as in Remark 7.12 shows that in general the clique-width $cw(I(\mathfrak{A}))$ of $I(\mathfrak{A})$ is not bounded by a function of $cw(\mathfrak{A})$.

Using Theorem 7.4 and Proposition 7.14 we get

Theorem 7.15. *Let $\phi \in CGSOL(\tau)$ be such that all its models are of tree-width at most k. Then the spectrum of ϕ is ultimately periodic.*

8. Conclusions and open problems

We have shown that the many-sorted spectra of $CMSOL$-sentences ϕ are ultimately s-periodic provided the models of ϕ are all of bounded patch-width, a generalization of tree-width and clique-width known from graph theory.

Our proofs are quite non-constructive, although very unfeasible bounds for the ultimate periodicity can be computed. These bounds depend only on the vocabulary τ, the quantifier rank q and the width under consideration, k, but are the same for all $\phi \in CMSOL^q(\tau)$.

Problem 8.1. *Find better estimates, exploiting features of ϕ as well.*

We have also shown that every ultimately s-periodic set $X \subseteq \mathbb{N}^s$ can be realized as an s-sorted FOL-spectrum with all large enough models of given tree-width exactly k.

We could think of sharpening this. Assume ϕ is an $FOL(\tau)$-sentence with s-ultimately periodic spectrum. Is there a finite axiomatizable theory interpretable in the deductive closure of ϕ with the same spectrum, and all its models of bounded tree-width (clique-width or patch-width)? More precisely:

Problem 8.2. *Assume ϕ is an $FOL(\tau)$-sentence with an s-ultimately periodic spectrum. Is there a vocabulary $\sigma = \{S_1, S_2, \ldots, S_m\}$ with S_i of arity ρ_i, an $FOL(\sigma)$-sentence ψ, and $FOL(\tau)$-formulas $\theta_i(x_1, \ldots, x_{\rho_i})$ such that*

(i) *$mspec_s(\phi) = mspec_s(\psi)$;*

(ii) *All models of ψ are of tree-width at most k for some k depending on ϕ;*

(iii) *$\phi \models \psi |_{S_i}^{\theta_i}$.*

Here the θ_i's define the interpretation and $\psi |_{S_i}^{\theta_i}$ is the result of substituting the S_i's by the θ_i's with appropriate choices of free variables.

For one-sorted spectra we have that the complement of an ultimately periodic set $X \subseteq \mathbb{N}$ is also ultimately periodic. Hence Asser's question has a positive answer for FOL-sentences ϕ where all its models are of bounded patch-width.

For semilinear sets in $\mathbb{N}^s, s \geq 2$, the following was shown by S. Ginsburg and E.H. Spanier in [GS66]:

Proposition 8.3. *The family of semilinear sets in $\mathbb{N}^s$ is closed under finite boolean operations.*

Hence we have, using Proposition 8.3 together with Theorem 7.5:

Corollary 8.4. *The complement $\mathbb{N}^s - mspec(\phi)$ of an s-sorted spectrum of an FOL-sentence ϕ, where all its models are of patch-width at most k, is also a many-sorted spectrum. In fact it may be taken from an FOL-sentence where all the models are of tree-width at most 1.*

However, without the assumption on patch-width, the following remains open:

Problem 8.5. *Is the complement of a many-sorted spectrum of an FOL-sentence also a many-sorted spectrum of an FOL-sentence?*

Finally, we may want to count the number $N_\phi(n)$ of labeled models of ϕ of fixed cardinality n, rather than look at the spectrum. The remarkable Specker-Blatter Theorem, [Spe88], says that for every $m \in \mathbb{N}$ the function $N_\phi(n)$ is ultimately periodic modulo m, provided $\phi \in MSOL(\tau)$ where τ contains only unary and binary relation symbols.

Theorem 8.6 (Specker and Blatter, 1981)**.** *Let $\phi \in MSOL(\tau)$ where τ contains only unary and binary relation symbols. For every $m \in \mathbb{N}$, there are $d_m, a_j^{(m)} \in \mathbb{N}$ such that the function N_ϕ satisfies the linear recurrence relation $N_\phi(n) \equiv \sum_{j=1}^{d_m} a_j^{(m)} N_\phi(n-j) \pmod{m}$, and hence is ultimately periodic modulo m.*

Fischer showed that this does not hold even for FOL-sentences if we allow quaternary relation symbols, [Fis03]. In [FM03b] we showed that it does hold for $\phi \in CMSOL$ for arbitrary relational vocabularies provided the relations have all bounded degree. For structures of size n of tree-width at most k the number of hyperedges is bounded by a function $O(n)$. This is not true for bounded clique-width. Hence we ask

Problem 8.7. *Does the Specker-Blatter Theorem hold for $\phi \in CMSOL$ provided all its models are of tree-width at most k?*

S. Shelah in [She04] looks at decomposability conditions to obtain an analysis of the spectrum. We give here the definitions which allow us a comparison of the results.

Definition 8.8. *Let $\mathfrak{A}$ be a τ-structure and $k \leq m \in \mathbb{N}$.*

(i) $\mathfrak{A}$ *is* (k,m)*-decomposable, if it has size at least* $k+2$ *and there are* τ*-structures* $\mathfrak{B}, \mathfrak{C}$*, both of size at least* m*, such that* $\mathfrak{A} = \mathfrak{B} \cup \mathfrak{C}$ *and* $\mathfrak{B} \cap \mathfrak{C}$ *has at most* k *elements.*

(ii) *A class* K *of* τ*-structures is* k-decomposable[10]*, or is a* $D_\tau(k)$*-class, if for every* $m \in \mathbb{N}$ *there is an* $n \in \mathbb{N}$ *such that every* $\mathfrak{A} \in K$ *of size at least* n *is* (k,m)*-decomposable.*

Observation 8.9. *Clearly, a* k*-tree decomposition of a structure* $\mathfrak{A}$ *of size at least* $k+2$ *can be used to show that* $\mathfrak{A}$ *is* $(k,k+2)$*-decomposable. Furthermore, if* K *is in* $TW(k)$ *and has arbitrarily large finite structures then* K *is in* $D(k+1)$*.*

S. Shelah in [She04] studies the spectrum of $\phi \in MSOL(\tau)$ where all the models are in $D_\tau(k)$ for some k. However, he does not prove ultimate periodicity for this case, but does gain some structural information concerning the gaps in the spectrum.

Definition 8.10. *Let* $\phi \in CMSOL$*. We define the function*

$$gap_\phi(n) = Min_t\{t \in spec(\phi) : t \geq n\}$$

Theorem 8.11 (Shelah 2003)**.** *If* $\phi \in MSOL(\tau)$ *such that its finite models are in* $D_\tau(k)$*, and* $\alpha > 0$ *is a real number, then for* $n \in \mathbb{N}$ *large enough*

$$\frac{gap_\phi(n)}{n+1} < (1+\alpha).$$

In comparison to Shelah's Theorem, we get the following immediately from Lemma, 4.1, Lemma 5.4 and the proof of Theorem 6.1 for classes of bounded patch-width.

Corollary 8.12. *If* $\phi \in MSOL(\tau)$ *such that* ϕ *has arbitrarily large finite models in* $PW_\tau(k)$*, then there is a real* $\beta > 0$ *such that for* $n \in \mathbb{N}$ *large enough*

$$\frac{gap_\phi(n)}{n+1} < \frac{\beta}{n}.$$

Problem 8.13. *What is the exact relationship between the class* $PW_\tau(k)$ *of* τ*-structures of patch-width at most* k *and the corresponding class* $D_\tau(k)$*?*

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[10]In the first posted version of [She04] this was called *weakly* k*-decomposable.*

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